

Fusion of irreducible modules in $\mathcal{WLM}(p, p')$

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Abstract

Based on symmetry principles, we derive a fusion algebra generated from repeated fusions of the irreducible modules appearing in the \mathcal{W} -extended logarithmic minimal model $\mathcal{WLM}(p, p')$. In addition to the irreducible modules themselves, closure of the commutative and associative fusion algebra requires the participation of a variety of reducible yet indecomposable modules. We conjecture that this fusion algebra is the same as the one obtained by application of the Nahm-Gaberdiel-Kausch algorithm and find that it reproduces the known such results for $\mathcal{WLM}(1, p')$ and $\mathcal{WLM}(2, 3)$. For $p > 1$, this fusion algebra does not contain a unit. Requiring that the spectrum of modules is invariant under a natural notion of conjugation, however, introduces an additional $(p-1)(p'-1)$ reducible yet indecomposable rank-1 modules, among which the identity is found, still yielding a well-defined fusion algebra. In this greater fusion algebra, the aforementioned symmetries are generated by fusions with the three irreducible modules of conformal weights $\Delta_{kp-1,1}$, $k = 1, 2, 3$. We also identify polynomial fusion rings associated with our fusion algebras.

1 Introduction

We consider the infinite series of Yang-Baxter integrable logarithmic minimal models $\mathcal{LM}(p, p')$ [1, 2] viewed in the \mathcal{W} -extended picture [3, 4, 5, 6] where they are denoted by $\mathcal{WLM}(p, p')$. The extension is believed to be with respect to the $\mathcal{W} = \mathcal{W}_{p,p'}$ symmetry algebra of [7], and we are considering the models in their continuum scaling limits. An object of great interest is the fusion algebra, here denoted by $\text{Irr}[\mathcal{WLM}(p, p')]$, generated from repeated fusions of the $2pp' + \frac{1}{2}(p-1)(p'-1)$ irreducible modules in $\mathcal{WLM}(p, p')$. For $p > 1$, we do not have boundary conditions associated with all of these modules and are therefore incapable of determining the complete set of fusion rules within our lattice approach based on Cardy's picture [8, 9]. On the other hand, this fusion algebra is believed to be obtainable using the Nahm-Gaberdiel-Kausch algorithm [10, 11]. The application of this algorithm is very tedious, however, even for $p = 1$ and in the case $\mathcal{WLM}(2, 3)$, so the goal here is to access the fusion rules by other means.

First, we construct a fusion algebra whose spectrum of modules contains all the irreducible modules in addition to $8pp' - 6p - 6p' + 4$ reducible yet indecomposable modules of which $2(p-1)(p'-1)$ are of rank 1, $4pp' - 2p - 2p'$ are of rank 2, and $2(p-1)(p'-1)$ are of rank 3. This fusion algebra is obtained from the fundamental fusion algebra $\text{Fund}[\mathcal{WLM}(p, p')]$, defined in [6], as the minimal extension thereof which is invariant under a particular triplet of symmetries. Here, we say that a fusion algebra \mathcal{A} with fusion multiplication \otimes is invariant under \mathcal{O} , or simply \mathcal{O} -symmetric, if

$$\mathcal{O}[\mathcal{R}] \otimes \mathcal{R}' = \mathcal{R} \otimes \mathcal{O}[\mathcal{R}'] = \mathcal{O}[\mathcal{R} \otimes \mathcal{R}'], \quad \forall \mathcal{R}, \mathcal{R}' \in \mathcal{J} \quad (1.1)$$

where \mathcal{O} is a map from and to the spectrum or set \mathcal{J} of modules underlying the fusion algebra. The spectrum of this extension of $\text{Fund}[\mathcal{WLM}(p, p')]$ is also invariant under a natural notion of conjugation. The extended fusion algebra itself is therefore denoted by $\text{Conj}[\mathcal{WLM}(p, p')]$. We prove that the

three symmetries of $\text{Conj}[\mathcal{WLM}(p, p')]$ are generated by fusion with the three irreducible modules of conformal weights $\Delta_{kp-1,1} = \Delta_{1,kp'-1}$, $k = 1, 2, 3$, where $\Delta_{\rho,\sigma}$ is given by the usual Kac formula. For critical percolation in the \mathcal{W} -extended picture $\mathcal{WLM}(2, 3)$, these conformal weights are $\Delta_{1,1} = 0$, $\Delta_{3,1} = 2$ and $\Delta_{5,1} = 7$.

As a subalgebra of $\text{Conj}[\mathcal{WLM}(p, p')]$, we identify the algebra generated by repeated fusions of the irreducible modules. We conjecture that this fusion algebra is indeed $\text{Irr}[\mathcal{WLM}(p, p')]$ and note that it is obtained from $\text{Conj}[\mathcal{WLM}(p, p')]$ by omitting $(p-1)(p'-1)$ of the reducible yet indecomposable rank-1 modules. That our proposal for $\mathcal{WLM}(1, p')$ yields the known results [12, 13, 3] is ensured by construction, while we have verified that it also reproduces the very recent results for $\mathcal{WLM}(2, 3)$ [14].

For $p > 1$, the fusion algebra $\text{Irr}[\mathcal{WLM}(p, p')]$ does *not* have a unit nor is its spectrum invariant under conjugation. The minimal extension, whose spectrum *is* conjugation invariant, is the fusion algebra $\text{Conj}[\mathcal{WLM}(p, p')]$, and this algebra *does* contain an identity. In this setting, conjugation invariance of the spectrum thus implies the existence of an identity. The paper [14] on $\mathcal{WLM}(2, 3)$ is actually focussed on such a conjugation-invariant spectrum, and we have verified that $\text{Conj}[\mathcal{WLM}(2, 3)]$ indeed corresponds to their results.

We also identify a polynomial fusion ring isomorphic to $\text{Conj}[\mathcal{WLM}(p, p')]$. For $p = 1$, where $\text{Conj}[\mathcal{WLM}(1, p')] = \text{Fund}[\mathcal{WLM}(1, p')]$, this was already done in [6] and involved a quotient polynomial ring with two generators. For $p > 1$, on the other hand, we find that the sought-after quotient polynomial ring has *five* generators corresponding to the two fundamental representations of $\text{Fund}[\mathcal{WLM}(p, p')]$ and the three symmetry-generating irreducible modules $\mathcal{W}(\Delta_{kp-1,1})$, $k = 1, 2, 3$. The fusion algebra $\text{Irr}[\mathcal{WLM}(p, p')]$ is isomorphic to a subring thereof.

Many subalgebras and quotients can be identified of the various fusion algebras discussed here. In [15], a general framework is outlined within which it makes sense to discuss rings of equivalence classes of fusion-algebra generators. Grothendieck-like rings, as the one generated by the $2pp'$ generators $\mathcal{K}_{r,s}^{\pm}$ in [7], arise as particularly interesting cases obtained by elevating character identities to equivalence relations between the corresponding fusion generators.

Notation and terminology

Unless otherwise specified, we let

$$\kappa, \kappa' \in \mathbb{Z}_{1,2}, \quad r \in \mathbb{Z}_{1,p}, \quad s \in \mathbb{Z}_{1,p'}, \quad a, a' \in \mathbb{Z}_{1,p-1}, \quad b, b' \in \mathbb{Z}_{1,p'-1}, \quad \alpha \in \mathbb{Z}_{0,p-1}, \quad \beta \in \mathbb{Z}_{0,p'-1} \quad (1.2)$$

where

$$\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n, m], \quad n, m \in \mathbb{Z} \quad (1.3)$$

denotes the set of integers from n to m , both included. By an expression like $\kappa \cdot \kappa'$, we mean $1 \cdot 1 = 2 \cdot 2 = 1$ or $1 \cdot 2 = 2 \cdot 1 = 2$. As a simplified notation for a set of elements with labels of the form (1.2), we write

$$\{f_{\kappa,a}, g_{\alpha}^{0,s}\} = \{f_{\kappa,a}; \kappa \in \mathbb{Z}_{1,2}, a \in \mathbb{Z}_{1,p-1}\} \cup \{g_{\alpha}^{0,s}; \alpha \in \mathbb{Z}_{0,p-1}, s \in \mathbb{Z}_{1,p'}\} \quad (1.4)$$

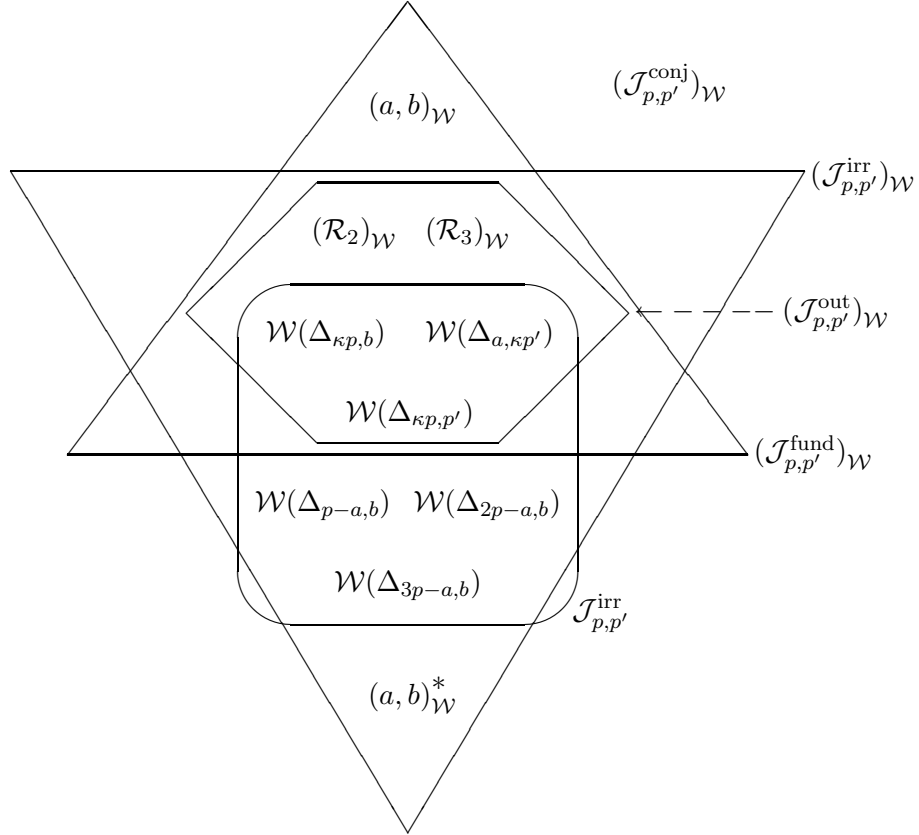
for example. The two terms *representation* and *module* are often used interchangeably when the discussion is on modules. Here, we use the term module.

Sets of indecomposable modules and their intersection diagram

To assist the reader, the various sets of indecomposable modules are summarized here. Their intersection diagram appears in (1.6) below. As convenient abbreviations, in (1.6), we let $(\mathcal{R}_2)_{\mathcal{W}}$ and $(\mathcal{R}_3)_{\mathcal{W}}$ denote the sets of indecomposable modules of rank 2 and 3, respectively, while $\mathcal{W}(\Delta_{p-a,b})$ represents the set $\{\mathcal{W}(\Delta_{p-a,b}); a \in \mathbb{Z}_{1,p-1}, b \in \mathbb{Z}_{1,p'-1}\}$, and so on. If \mathcal{J} is a subset of \mathcal{J}' , the fusion algebra $\langle \mathcal{J} \rangle$ generated from \mathcal{J} is a subalgebra of the fusion algebra $\langle \mathcal{J}' \rangle$ generated from \mathcal{J}' . It is noted that the fusions may generate indecomposable modules not listed explicitly in the two sets.

Now, the set of indecomposable modules associated with boundary conditions is denoted by $(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$ and is represented in (1.6) by the interior of the hexagon. The set of indecomposable modules appearing in the fundamental fusion algebra is denoted by $(\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}$ and is represented by the interior of the upward-pointing triangle. The set of irreducible modules is denoted by $\mathcal{J}_{p,p'}^{\text{irr}}$ and is represented by the interior of the soft-cornered square. The set of indecomposable modules generated by repeated fusions of the irreducible modules is denoted by $(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}$ and is represented by the interior of the downward-pointing triangle. The total set of indecomposable modules considered here is denoted by $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$. The cardinalities of these sets are

$$\begin{aligned}
|(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}| &= 6pp' - 2p - 2p', & |(\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}| &= 7pp' - 3p - 3p' + 1 \\
|J_{p,p'}^{\text{irr}}| &= 2pp' + \frac{1}{2}(p-1)(p'-1), & |(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}| &= 4p + 4p' - 6 + \frac{19}{2}(p-1)(p'-1) \\
|(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}| &= 4p + 4p' - 6 + \frac{21}{2}(p-1)(p'-1)
\end{aligned} \tag{1.5}$$



(1.6)

2 Fundamental fusion algebra of $\mathcal{WLM}(p, p')$

A logarithmic minimal model $\mathcal{LM}(p, p')$ is defined [1, 2] for every coprime pair of positive integers $p < p'$. The model has central charge

$$c = 1 - 6 \frac{(p' - p)^2}{pp'} \tag{2.1}$$

and conformal weights

$$\Delta_{\rho,\sigma} = \frac{(\rho p' - \sigma p)^2 - (p' - p)^2}{4pp'}, \quad \rho, \sigma \in \mathbb{N} \tag{2.2}$$

Its \mathcal{W} -extension $\mathcal{WLM}(p, p')$ is discussed in [3, 4, 5, 6] and briefly reviewed in the following.

2.1 Modules associated with boundary conditions

The indecomposable modules in $\mathcal{WLM}(p, p')$ which can be associated with Yang-Baxter integrable boundary conditions on the strip lattice and \mathcal{W} -invariant boundary conditions in the continuum scaling limit, were identified in [4, 5] by extending constructions in [3] pertaining to the case $p = 1$. The set of these modules is given by

$$(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} = \{\mathcal{W}(\Delta_{\kappa p, b}), \mathcal{W}(\Delta_{a, \kappa p'}), \mathcal{W}(\Delta_{\kappa p, p'}), (\mathcal{R}_{\kappa p, s}^{a, 0})_{\mathcal{W}}, (\mathcal{R}_{r, \kappa p'}^{0, b})_{\mathcal{W}}, (\mathcal{R}_{\kappa p, p'}^{a, b})_{\mathcal{W}}\} \quad (2.3)$$

and is of cardinality

$$|(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}| = 6pp' - 2p - 2p' \quad (2.4)$$

Here, we have adopted the notation of [14] denoting a \mathcal{W} -irreducible module of conformal weight Δ by $\mathcal{W}(\Delta)$. Thus, there are $2p + 2p' - 2$ irreducible (hence indecomposable rank-1) modules

$$\{\mathcal{W}(\Delta_{\kappa p, s}), \mathcal{W}(\Delta_{r, \kappa p'})\} \quad (2.5)$$

where the two modules $\mathcal{W}(\Delta_{\kappa p, p}) = \mathcal{W}(\Delta_{p, \kappa p'})$ are listed twice, in addition to $4pp' - 2p - 2p'$ indecomposable rank-2 modules

$$\{(\mathcal{R}_{\kappa p, s}^{a, 0})_{\mathcal{W}}, (\mathcal{R}_{r, \kappa p'}^{0, b})_{\mathcal{W}}\} \quad (2.6)$$

and $2(p-1)(p'-1)$ indecomposable rank-3 modules

$$\{(\mathcal{R}_{\kappa p, \kappa' p'}^{a, b})_{\mathcal{W}}\} \quad \text{subject to} \quad (\mathcal{R}_{p, 2p'}^{a, b})_{\mathcal{W}} \equiv (\mathcal{R}_{2p, p'}^{a, b})_{\mathcal{W}} \quad \text{and} \quad (\mathcal{R}_{2p, 2p'}^{a, b})_{\mathcal{W}} \equiv (\mathcal{R}_{p, p'}^{a, b})_{\mathcal{W}} \quad (2.7)$$

The fusion algebra of these modules [5, 6]

$$\text{Out}[\mathcal{WLM}(p, p')] = \langle (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \rangle \quad (2.8)$$

is given explicitly in Appendix A.1 as (A.6) through (A.11) and is both associative and commutative. There is no unit or identity for $p > 1$, while, for $p = 1$, the irreducible module $\mathcal{W}(\Delta_{1,1})$ is the identity.

In [5], it was conjectured that every indecomposable rank-2 module has an embedding pattern of one of the types

$$\begin{array}{ccc} & \mathcal{W}(\Delta_v) & \\ \swarrow & & \searrow \\ \mathcal{E}(\Delta_h, \Delta_v) : \mathcal{W}(\Delta_h) & \longleftrightarrow & \mathcal{W}(\Delta_h) \\ \swarrow & & \searrow \\ & \mathcal{W}(\Delta_v) & \end{array} \quad \begin{array}{ccc} & \mathcal{W}(\Delta_v) & \\ \swarrow & & \searrow \\ \mathcal{E}(\Delta_h, \Delta_v; \Delta_c) : \mathcal{W}(\Delta_h) & \longleftrightarrow & \mathcal{W}(\Delta_h) \\ & \swarrow \quad \searrow & \\ & \mathcal{W}(\Delta_c) & \\ & \swarrow \quad \searrow & \\ & \mathcal{W}(\Delta_v) & \end{array} \quad (2.9)$$

where the horizontal arrows indicate the non-diagonal action of the Virasoro mode L_0 . Specifically, the indecomposable rank-2 modules (2.6) are believed to enjoy the embedding patterns

$$\begin{aligned} (\mathcal{R}_{p, b}^{a, 0})_{\mathcal{W}} &\sim \mathcal{E}(\Delta_{p+a, b}, \Delta_{3p-a, b}; \Delta_{p-a, b}), & (\mathcal{R}_{a, p'}^{0, b})_{\mathcal{W}} &\sim \mathcal{E}(\Delta_{a, p'+b}, \Delta_{a, 3p'-b}; \Delta_{a, p'-b}) \\ (\mathcal{R}_{p, p'}^{a, 0})_{\mathcal{W}} &\sim \mathcal{E}(\Delta_{p+a, p'}, \Delta_{3p-a, p'}), & (\mathcal{R}_{p, p'}^{0, b})_{\mathcal{W}} &\sim \mathcal{E}(\Delta_{p, p'+b}, \Delta_{p, 3p'-b}) \\ (\mathcal{R}_{2p, s}^{a, 0})_{\mathcal{W}} &\sim \mathcal{E}(\Delta_{2p+a, s}, \Delta_{2p-a, s}), & (\mathcal{R}_{r, 2p'}^{0, b})_{\mathcal{W}} &\sim \mathcal{E}(\Delta_{r, 2p'+b}, \Delta_{r, 2p'-b}) \end{aligned} \quad (2.10)$$

In [5], it was also conjectured that the indecomposable rank-3 modules (2.7) have embedding structures described by the patterns in (2.9), namely

$$(\mathcal{R}_{\kappa p, p'}^{a, b})_{\mathcal{W}} \sim \mathcal{E}\left((\mathcal{R}_{\kappa p, p'-b}^{a, 0})_{\mathcal{W}}, (\mathcal{R}_{(3-\kappa)p, b}^{a, 0})_{\mathcal{W}}\right) \sim \mathcal{E}\left((\mathcal{R}_{p-a, \kappa p'}^{0, b})_{\mathcal{W}}, (\mathcal{R}_{a, (3-\kappa)p'}^{0, b})_{\mathcal{W}}\right) \quad (2.11)$$

where the irreducible modules $\mathcal{W}(\Delta_h)$ and $\mathcal{W}(\Delta_v)$ have been replaced by indecomposable rank-2 modules.

2.2 Supplementary modules

In [6], based on algebraic arguments, we suggested to supplement the set of indecomposable modules (2.3) by the reducible yet indecomposable rank-1 modules

$$\{(a, b)_{\mathcal{W}}\} \quad (2.12)$$

The cardinality of the disjoint union

$$(\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}} = \{(a, b)_{\mathcal{W}}\} \cup (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \quad (2.13)$$

is therefore given by

$$|(\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}| = (p-1)(p'-1) + |(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}| = 7pp' - 3p - 3p' + 1 \quad (2.14)$$

We also argued that the embedding pattern of $(a, b)_{\mathcal{W}}$ is of the form

$$\begin{array}{c} \mathcal{W}(\Delta_{2p-a,b}) \\ \swarrow \\ \mathcal{W}(\Delta_{a,b}) \end{array} \quad (2.15)$$

implying the short exact sequence

$$0 \rightarrow \mathcal{W}(\Delta_{2p-a,b}) \rightarrow (a, b)_{\mathcal{W}} \rightarrow \mathcal{W}(\Delta_{a,b}) \rightarrow 0 \quad (2.16)$$

The algebraic extension (2.13) of the set of indecomposable modules (2.3) was shown in [6] to yield a well-defined fusion algebra called the fundamental fusion algebra and denoted by

$$\text{Fund}[\mathcal{WLM}(p, p')] = \langle (\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}} \rangle \quad (2.17)$$

The underlying fusion rules are all listed in Appendix A.1. The algebra is generated from repeated fusions of the two ‘fundamental representations’ $(2, 1)_{\mathcal{W}}$ and $(1, 2)_{\mathcal{W}}$ (strictly speaking, in addition to the identity $(1, 1)_{\mathcal{W}}$)

$$\text{Fund}[\mathcal{WLM}(p, p')] = \langle (1, 1)_{\mathcal{W}}, (2, 1)_{\mathcal{W}}, (1, 2)_{\mathcal{W}} \rangle \quad (2.18)$$

From [6], based on an explicit inspection of the fusion rules, we know that $\text{Out}[\mathcal{WLM}(p, p')]$ is an ideal of $\text{Fund}[\mathcal{WLM}(p, p')]$. Since the set (2.12) is empty for $p = 1$, we note that

$$\text{Fund}[\mathcal{WLM}(1, p')] = \text{Out}[\mathcal{WLM}(1, p')] \quad (2.19)$$

3 Fusion of irreducible modules in $\mathcal{WLM}(p, p')$

3.1 Modules

It is the same set of irreducible modules which appears as subfactors of the indecomposable modules in (2.3) as in (2.13). This set is given by

$$\mathcal{J}_{p,p'}^{\text{irr}} = \{\mathcal{W}(\Delta_{\rho,\sigma}); \rho p' \geq \sigma p, \rho \in \mathbb{Z}_{1,3p-1}, \sigma \in \mathbb{Z}_{1,p'}\} \quad (3.1)$$

and we recall the simple identities

$$\Delta_{a,b} = \Delta_{a+kp, b+kp'}, \quad \Delta_{a, kp'-b} = \Delta_{kp-a, b}, \quad k \in \mathbb{Z} \quad (3.2)$$

allowing a great deal of freedom in the labeling of the conformal weights. As a matter of convention, we have chosen the labeling indicated in (3.1). This set of irreducible modules also appears in [7] and has cardinality

$$|J_{p,p'}^{\text{irr}}| = 2pp' + \frac{1}{2}(p-1)(p'-1) \quad (3.3)$$

For $p > 1$, the set (3.1) is larger than the set of irreducible modules (2.5) appearing as generators in the fundamental fusion algebra. It is thus natural to try to understand the fusion algebra resulting from repeated fusions of the irreducible modules (3.1) and to determine the set of modules required to ensure closure of this fusion algebra. Even though the set (2.5) is a subset of (3.1), there is, a priori, no need for the fundamental fusion algebra to be a subalgebra of this fusion algebra since the former is generated by the two fundamental representations which *may not*, after all, arise from repeated fusions of the irreducible modules in (3.1). Indeed, the fusion algebra $\text{Irr}[\mathcal{WLM}(p, p')]$, to be discussed below, does *not* contain the fundamental fusion algebra as a subalgebra, while the fusion algebra $\text{Conj}[\mathcal{WLM}(p, p')]$, also to be discussed below, *does*.

A complicating factor for $p > 1$ is that we do not have boundary conditions associated with all of the irreducible modules in (3.1) (only with the ones appearing in (2.5)) and are therefore incapable of determining the complete set of fusion rules within our lattice approach. On the other hand, the sought-after fusion algebra is believed to be obtainable using the Nahm-Gaberdiel-Kausch algorithm [10, 11]. The application of this algorithm is very tedious, however, even for $p = 1$ and in the case $\mathcal{WLM}(2, 3)$, so an alternative approach to the fusion rules is certainly welcome. Our proposal below is to use symmetry principles, and we have verified, as we will discuss, that our conjectured fusion algebras indeed reproduce the known results obtained using the algorithm.

In preparation for the discussion of fusion rules, we introduce the $(p-1)(p'-1)$ reducible yet indecomposable rank-1 modules

$$\{(a, b)_{\mathcal{W}}^*\} \quad (3.4)$$

whose embedding patterns

$$\begin{array}{ccc} \mathcal{W}(\Delta_{2p-a,b}) & \searrow & \\ & & \mathcal{W}(\Delta_{a,b}) \end{array} \quad (3.5)$$

imply the short exact sequences

$$0 \rightarrow \mathcal{W}(\Delta_{a,b}) \rightarrow (a, b)_{\mathcal{W}}^* \rightarrow \mathcal{W}(\Delta_{2p-a,b}) \rightarrow 0 \quad (3.6)$$

It follows immediately that

$$\chi[(a, b)_{\mathcal{W}}^*](q) = \chi[(a, b)_{\mathcal{W}}](q) \quad (3.7)$$

where the characters $\chi[(a, b)_{\mathcal{W}}](q)$ are discussed in [6] alongside the characters of all the other modules appearing in the fusion algebra.

It may seem surprising that we are introducing the contragredient modules $(a, b)_{\mathcal{W}}^*$. To motivate their appearance, we briefly consider fusion of the underlying Virasoro modules in $\mathcal{LM}(p, p')$. Details thereof may be found in [16], in particular in the case $\mathcal{LM}(2, 3)$, and are obtained using the Nahm-Gaberdiel-Kausch algorithm. Let us denote by $\mathcal{V}(\Delta)$ the irreducible Virasoro module of conformal weight Δ . A careful re-examination of the fusion $\mathcal{V}(2) \otimes \mathcal{V}(2)$ in $\mathcal{LM}(2, 3)$ reveals that a natural but incorrect identification was made in [16]. This was also observed in [14]. The correct fusion rule reads

$$\mathcal{V}(2) \otimes \mathcal{V}(2) = (1, 1)^* \quad (3.8)$$

where $(1, 1)^*$ is the indecomposable module contragredient to the indecomposable identity module $(1, 1)$

$$0 \rightarrow \mathcal{V}(2) \rightarrow (1, 1) \rightarrow \mathcal{V}(0) \rightarrow 0, \quad 0 \rightarrow \mathcal{V}(0) \rightarrow (1, 1)^* \rightarrow \mathcal{V}(2) \rightarrow 0 \quad (3.9)$$

Continuing this analysis also sees the introduction of the two indecomposable Virasoro modules $(1, 2)$ and $(1, 2)^*$ corresponding to the short exact sequences

$$0 \rightarrow \mathcal{V}(1) \rightarrow (1, 2) \rightarrow \mathcal{V}(0) \rightarrow 0, \quad 0 \rightarrow \mathcal{V}(0) \rightarrow (1, 2)^* \rightarrow \mathcal{V}(1) \rightarrow 0 \quad (3.10)$$

In $\mathcal{LM}(p, p')$, this generalizes to pairs (a, b) and $(a, b)^*$ whose \mathcal{W} -extended counterparts in $\mathcal{WLM}(p, p')$ we have denoted by $(a, b)_{\mathcal{W}}$ and $(a, b)_{\mathcal{W}}^*$, respectively.

Returning to the preparations, we also introduce the sets

$$(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} = (\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}} \cup \{(a, b)_{\mathcal{W}}\}, \quad (\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}} = J_{p,p'}^{\text{irr}} \cup \{(a, b)_{\mathcal{W}}^*, (\mathcal{R}_{\kappa p, s}^{a,0})_{\mathcal{W}}, (\mathcal{R}_{r, \kappa p'}^{0,b})_{\mathcal{W}}, (\mathcal{R}_{\kappa p, p'}^{a,b})_{\mathcal{W}}\} \quad (3.11)$$

as disjoint unions. Their cardinalities are thus

$$|(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}| = |(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}| + (p-1)(p'-1), \quad |(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}| = 4p + 4p' - 6 + \frac{19}{2}(p-1)(p'-1) \quad (3.12)$$

and will appear as the dimensions of two of the fusion algebras to be discussed. The notation $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ and $(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}$ will become clear in the following.

3.2 Spectrum maps

To facilitate the description of the fusion algebra generated from repeated fusions of the irreducible modules (3.1), we now introduce some maps from $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ to itself. We first extend the use of $*$ in (3.4) to an involution, here denoted by \mathcal{C} and referred to as *conjugation*, on the entire set of modules $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ by

$$\mathcal{C}[(a, b)_{\mathcal{W}}] = (a, b)_{\mathcal{W}}^*, \quad \mathcal{C}[(a, b)_{\mathcal{W}}^*] = (a, b)_{\mathcal{W}}, \quad \mathcal{C}[\mathcal{R}] = \mathcal{R}, \quad \mathcal{R} \in (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \cup J_{p,p'}^{\text{irr}} \quad (3.13)$$

Since the embedding patterns (2.10) and (2.11) are invariant under reversal of the arrows, we see that the conjugation \mathcal{C} , as an operation on the embedding patterns, simply reverses the arrows. We note that this is trivially true when applied also to the irreducible modules.

We also introduce the map \mathcal{K} which, on $(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$, acts by $\kappa \leftrightarrow 2 \cdot \kappa = 3 - \kappa$ on the labeling of the modules as given in (2.3), vanishes on $\{\mathcal{W}(\Delta_{a,b})\}$, while its action on $\{(a, b)_{\mathcal{W}}, (a, b)_{\mathcal{W}}^*, \mathcal{W}(\Delta_{\kappa p+a,b})\}$ is described by the diagram

$$\begin{array}{ccccc} & & \mathcal{W}(\Delta_{2p-a,b}) & & \\ & & \downarrow \mathcal{K} & & \\ (a, b)_{\mathcal{W}} & \xrightarrow{\mathcal{K}} & \mathcal{W}(\Delta_{3p-a,b}) & \xleftarrow{\mathcal{K}} & (a, b)_{\mathcal{W}}^* \end{array} \quad (3.14)$$

We thus have

$$\begin{aligned} \mathcal{K}[(\mathcal{R}_{\kappa p, s}^{a,0})_{\mathcal{W}}] &= (\mathcal{R}_{(2 \cdot \kappa)p, s}^{a,0})_{\mathcal{W}}, & \mathcal{K}[(\mathcal{R}_{r, \kappa p'}^{0,b})_{\mathcal{W}}] &= (\mathcal{R}_{r, (2 \cdot \kappa)p'}^{0,b})_{\mathcal{W}}, & \mathcal{K}[(\mathcal{R}_{\kappa p, p'}^{a,b})_{\mathcal{W}}] &= (\mathcal{R}_{(2 \cdot \kappa)p, p'}^{a,b})_{\mathcal{W}} \\ \mathcal{K}[\mathcal{W}(\Delta_{\kappa p, s})] &= \mathcal{W}(\Delta_{(2 \cdot \kappa)p, s}), & \mathcal{K}[\mathcal{W}(\Delta_{\kappa p+a, p'})] &= \mathcal{W}(\Delta_{(2 \cdot \kappa)p+a, p'}), & \mathcal{K}[\mathcal{W}(\Delta_{a,b})] &= 0 \\ \mathcal{K}[(a, b)_{\mathcal{W}}] &= \mathcal{K}[(a, b)_{\mathcal{W}}^*] = \mathcal{K}[\mathcal{W}(\Delta_{2p-a,b})] = \mathcal{W}(\Delta_{3p-a,b}), & \mathcal{K}[\mathcal{W}(\Delta_{3p-a,b})] &= (a, b)_{\mathcal{W}}^* \end{aligned} \quad (3.15)$$

Lemma 1 The fusion algebra $\langle (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \rangle$ is \mathcal{K} -symmetric (in the sense of (1.1) with $\mathcal{O} = \mathcal{K}$ restricted to $\mathcal{J} = (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$).

Lemma 2 In the fusion algebra $\langle (\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}} \rangle$, we have

$$(a, b)_{\mathcal{W}} \otimes \mathcal{K}[\mathcal{Q}] = \mathcal{K}[(a, b)_{\mathcal{W}} \otimes \mathcal{Q}], \quad \mathcal{Q} \in (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \quad (3.16)$$

\circ	\mathcal{K}	\mathcal{L}	\mathcal{L}^2	\mathcal{M}	\mathcal{M}^2
\mathcal{K}	\mathcal{L}^2	\mathcal{K}	\mathcal{K}	0	0
\mathcal{L}	\mathcal{K}	\mathcal{L}^2	\mathcal{L}^2	0	0
\mathcal{L}^2	\mathcal{K}	\mathcal{L}^2	\mathcal{L}^2	0	0
\mathcal{M}	0	0	0	\mathcal{M}^2	\mathcal{M}
\mathcal{M}^2	0	0	0	\mathcal{M}	\mathcal{M}^2

Figure 1: Cayley table of the composition rules for $\mathcal{K}, \mathcal{L}, \mathcal{L}^2, \mathcal{M}, \mathcal{M}^2 \in \mathcal{B}$ for $p > 1$.

Lemma 1 follows by direct inspection of the fusion rules (A.6) though (A.11), while Lemma 2 follows by direct inspection of the fusion rules (A.3) through (A.5).

The map \mathcal{L} is defined by

$$\begin{aligned} \mathcal{L}[(a, b)_{\mathcal{W}}] &= \mathcal{W}(\Delta_{2p-a, b}), & \mathcal{L}[\mathcal{W}(\Delta_{2p-a, b})] &= (a, b)_{\mathcal{W}}^*, & \mathcal{L}[\mathcal{W}(\Delta_{a, b})] &= 0 \\ \mathcal{L}[\mathcal{R}] &= \mathcal{R}, & \mathcal{R} &\in (\mathcal{J}_{p, p'}^{\text{irr}})_{\mathcal{W}} \setminus \{\mathcal{W}(\Delta_{\kappa p-a, b})\} \end{aligned} \quad (3.17)$$

while the map \mathcal{M} is defined by

$$\begin{aligned} \mathcal{M}[(a, b)_{\mathcal{W}}] &= \mathcal{W}(\Delta_{p-a, b}), & \mathcal{M}[\mathcal{W}(\Delta_{a, b})] &= \mathcal{W}(\Delta_{p-a, b}) \\ \mathcal{M}[\mathcal{R}] &= 0, & \mathcal{R} &\in (\mathcal{J}_{p, p'}^{\text{irr}})_{\mathcal{W}} \setminus \{\mathcal{W}(\Delta_{a, b})\} \end{aligned} \quad (3.18)$$

Since \mathcal{L} and \mathcal{M} both act trivially on $(\mathcal{J}_{p, p'}^{\text{out}})_{\mathcal{W}}$, Lemma 1 and 2 obviously apply also when replacing \mathcal{K} by either \mathcal{L} or \mathcal{M} (recalling that $(\mathcal{J}_{p, p'}^{\text{out}})_{\mathcal{W}}$ generates an ideal of $\langle (\mathcal{J}_{p, p'}^{\text{fund}})_{\mathcal{W}} \rangle$). We note that the introduction of \mathcal{L} and \mathcal{M} is meaningless for $p = 1$.

Under composition, the maps \mathcal{K}, \mathcal{L} and \mathcal{M} generate a five-dimensional commutative algebra whose composition rules in the basis

$$\mathcal{B} = \{\mathcal{K}, \mathcal{L}, \mathcal{L}^2, \mathcal{M}, \mathcal{M}^2\}, \quad \mathcal{L}^2 = \mathcal{L} \circ \mathcal{L}, \quad \mathcal{M}^2 = \mathcal{M} \circ \mathcal{M} \quad (3.19)$$

are summarized in Figure 1. This algebra has no unit but can, of course, be extended straightforwardly by inclusion of the identity map \mathcal{I} on $(\mathcal{J}_{p, p'}^{\text{conj}})_{\mathcal{W}}$. We note that $\{\mathcal{M}, \mathcal{M}^2\}$, for example, generates an ideal. Another interesting observation is that \mathcal{L} *itself* does not appear as the result of composing any of the maps $\mathcal{K}, \mathcal{L}^2 = \mathcal{K} \circ \mathcal{K}, \mathcal{M}, \mathcal{M}^2$. We will return to this point in Section 3.5.

In partial summary, and with \mathcal{Q}_{κ} denoting a general element of $(\mathcal{J}_{p, p'}^{\text{out}})_{\mathcal{W}}$, the diagrams

$$\begin{array}{ccccc} & & \mathcal{W}(\Delta_{2p-a, b}) & & \\ & \nearrow \mathcal{L} & \downarrow \mathcal{K} & \nwarrow \mathcal{L} & \\ \mathcal{W}(\Delta_{p-a, b}) & \xleftarrow{\mathcal{M}} & (a, b)_{\mathcal{W}} & \xrightarrow{\mathcal{K}} & \mathcal{W}(\Delta_{3p-a, b}) & \xleftarrow{\mathcal{K}} & (a, b)_{\mathcal{W}}^* \\ & & \nwarrow \mathcal{L} & \nearrow \mathcal{L} & & \\ & & \mathcal{C} & & \end{array} \quad (3.20)$$

	\mathcal{Q}_κ	$(a, b)_\mathcal{W}$	$\mathcal{W}(\Delta_{p-a,b})$	$\mathcal{W}(\Delta_{2p-a,b})$	$\mathcal{W}(\Delta_{3p-a,b})$	$(a, b)_\mathcal{W}^*$
$\mathcal{C} \circ \mathcal{K}$	$\mathcal{Q}_{2 \cdot \kappa}$	$\mathcal{W}(\Delta_{3p-a,b})$	0	$\mathcal{W}(\Delta_{3p-a,b})$	$(a, b)_\mathcal{W}$	$\mathcal{W}(\Delta_{3p-a,b})$
$\mathcal{K} \circ \mathcal{C}$	$\mathcal{Q}_{2 \cdot \kappa}$	$\mathcal{W}(\Delta_{3p-a,b})$	0	$\mathcal{W}(\Delta_{3p-a,b})$	$(a, b)_\mathcal{W}^*$	$\mathcal{W}(\Delta_{3p-a,b})$
$\mathcal{C} \circ \mathcal{L}$	\mathcal{Q}_κ	$\mathcal{W}(\Delta_{2p-a,b})$	0	$(a, b)_\mathcal{W}$	$\mathcal{W}(\Delta_{3p-a,b})$	$(a, b)_\mathcal{W}$
$\mathcal{L} \circ \mathcal{C}$	\mathcal{Q}_κ	$(a, b)_\mathcal{W}^*$	0	$(a, b)_\mathcal{W}^*$	$\mathcal{W}(\Delta_{3p-a,b})$	$\mathcal{W}(\Delta_{2p-a,b})$
$\mathcal{C} \circ \mathcal{M}$	0	$\mathcal{W}(\Delta_{p-a,b})$	$\mathcal{W}(\Delta_{a,b})$	0	0	0
$\mathcal{M} \circ \mathcal{C}$	0	0	$\mathcal{W}(\Delta_{a,b})$	0	0	$\mathcal{W}(\Delta_{p-a,b})$

Figure 2: Table indicating the results of acting with $\mathcal{C} \circ \mathcal{O}$ and $\mathcal{O} \circ \mathcal{C}$, for $\mathcal{O} \in \{\mathcal{K}, \mathcal{L}, \mathcal{M}\}$, on the various types of modules.

and

$$\begin{aligned}
\mathcal{Q}_\kappa &\xleftrightarrow{\mathcal{K}} \mathcal{Q}_{2 \cdot \kappa} & \mathcal{W}(\Delta_{a,b}) &\xleftrightarrow{\mathcal{M}} \mathcal{W}(\Delta_{p-a,b}) \\
\mathcal{W}(\Delta_{a,b}) &\xrightarrow{\mathcal{K}, \mathcal{L}} 0 & \xleftarrow{\mathcal{M}} & \mathcal{R} \in (\mathcal{J}_{p,p'}^{\text{irr}})_\mathcal{W} \setminus \{\mathcal{W}(\Delta_{a,b})\}
\end{aligned} \tag{3.21}$$

depict the non-trivial actions of the maps $\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{C}$ on $(\mathcal{J}_{p,p'}^{\text{conj}})_\mathcal{W}$. As indicated in Figure 2, the conjugation \mathcal{C} does in general *not* commute with the three maps \mathcal{K}, \mathcal{L} and \mathcal{M} . Its inclusion in the algebra in Figure 1 would thus result in a *non-commutative* composition algebra. Commutativity is respected on $(\mathcal{J}_{p,p'}^{\text{out}})_\mathcal{W} \cup \{\mathcal{W}(\Delta_{a,b})\}$, though. In all instances, the fusion algebras to be discussed in the following *are* commutative.

3.3 Symmetries and fusion rules

The fusion algebra $\text{Conj}[\mathcal{WLM}(p, p')]$ to be discussed presently is constructed as an extension of the fundamental fusion algebra $\text{Fund}[\mathcal{WLM}(p, p')]$ whose fusion rules

$$\mathcal{R}_i \otimes \mathcal{R}_j = \bigoplus_k \mathcal{N}_{i,j}^k \mathcal{R}_k, \quad \mathcal{R}_i, \mathcal{R}_j, \mathcal{R}_k \in (\mathcal{J}_{p,p'}^{\text{fund}})_\mathcal{W}, \quad \mathcal{N}_{i,j}^k \in \mathbb{N}_0 \tag{3.22}$$

are given in [6] and recalled in Appendix A.1. By an extension of a fusion algebra \mathcal{A} , we simply mean a fusion algebra containing \mathcal{A} as a non-trivial subalgebra.

Proposition 1 Introducing $\text{Conj}[\mathcal{WLM}(p, p')] = \langle (\mathcal{J}_{p,p'}^{\text{conj}})_\mathcal{W} \rangle$ as an extension of the fundamental fusion algebra (3.22), by requiring it to be \mathcal{K} -, \mathcal{L} - and \mathcal{M} -symmetric, yields a unique fusion algebra. It is commutative and associative and has only non-negative integer fusion multiplicities. The module $(1, 1)_\mathcal{W}$ is the unit.

Proof The uniqueness is an immediate consequence of the structure of the diagram (3.20) where every module in $(\mathcal{J}_{p,p'}^{\text{conj}})_\mathcal{W} \setminus (\mathcal{J}_{p,p'}^{\text{fund}})_\mathcal{W}$ can be written as $\mathcal{O}[(a, b)_\mathcal{W}]$ for some $\mathcal{O} \in \mathcal{B}$ and some $(a, b)_\mathcal{W} \in (\mathcal{J}_{p,p'}^{\text{fund}})_\mathcal{W}$. With $\mathcal{O}, \mathcal{O}', \mathcal{O}'' \in \mathcal{B}$, associativity follows from

$$\begin{aligned}
(\mathcal{O}[(a, b)_\mathcal{W}] \otimes \mathcal{O}'[(a', b')_\mathcal{W}]) \otimes \mathcal{O}''[(a'', b'')_\mathcal{W}] &= \mathcal{O}''[\mathcal{O}' \circ \mathcal{O}[(a, b)_\mathcal{W} \otimes (a', b')_\mathcal{W}] \otimes (a'', b'')_\mathcal{W}] \\
&= \mathcal{O}'' \circ \mathcal{O}' \circ \mathcal{O}[(a, b)_\mathcal{W} \otimes (a', b')_\mathcal{W} \otimes (a'', b'')_\mathcal{W}] \\
&= \mathcal{O}[(a, b)_\mathcal{W}] \otimes (\mathcal{O}'[(a', b')_\mathcal{W}] \otimes \mathcal{O}''[(a'', b'')_\mathcal{W}])
\end{aligned} \tag{3.23}$$

Commutativity follows similarly. The fusion multiplicities are all taken from the set of fusion multiplicities appearing in (3.22) and are therefore non-negative integers. Since every module $\mathcal{R} \in (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ can be written as $\mathcal{R} = \mathcal{O}[\mathcal{F}]$ for some $\mathcal{O} \in \mathcal{B} \cup \{I\}$ and $\mathcal{F} \in (\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}$, the unital property of $(1,1)_{\mathcal{W}}$ follows from

$$(1,1)_{\mathcal{W}} \otimes \mathcal{R} = (1,1)_{\mathcal{W}} \otimes \mathcal{O}[\mathcal{F}] = \mathcal{O}[(1,1)_{\mathcal{W}} \otimes \mathcal{F}] = \mathcal{O}[\mathcal{F}] = \mathcal{R} \quad (3.24)$$

□

As an immediate consequence of Proposition 1, we see that

$$\mathcal{O}[\mathcal{R}] \otimes \mathcal{O}'[\mathcal{R}'] = \mathcal{O}'[\mathcal{R}] \otimes \mathcal{O}[\mathcal{R}'], \quad \mathcal{O}, \mathcal{O}' \in \mathcal{B}, \quad \mathcal{R}, \mathcal{R}' \in (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \quad (3.25)$$

Combined with the composition algebra of Figure 1, it also follows that, in addition to (3.22), the fusion rules underlying $\text{Conj}[\mathcal{WLM}(p,p')]$ are given by¹

$$\begin{aligned} \mathcal{W}(\Delta_{2p-a,b}) \otimes \mathcal{Q}_{\kappa} &= (a,b)_{\mathcal{W}} \otimes \mathcal{Q}_{\kappa}, & \mathcal{W}(\Delta_{3p-a,b}) \otimes \mathcal{Q}_{\kappa} &= (a,b)_{\mathcal{W}} \otimes \mathcal{Q}_{2 \cdot \kappa} \\ \mathcal{W}(\Delta_{p-a,b}) \otimes \mathcal{Q}_{\kappa} &= 0, & (a,b)_{\mathcal{W}}^* \otimes \mathcal{Q}_{\kappa} &= (a,b)_{\mathcal{W}} \otimes \mathcal{Q}_{\kappa}, & \mathcal{Q}_{\kappa} &\in (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \end{aligned} \quad (3.26)$$

and (here written in ‘reverse order’)

$$\begin{aligned} 0 &= \mathcal{W}(\Delta_{p-a,b}) \otimes \mathcal{W}(\Delta_{2p-a',b'}) = \mathcal{W}(\Delta_{p-a,b}) \otimes \mathcal{W}(\Delta_{3p-a',b'}) \\ &= \mathcal{W}(\Delta_{p-a,b}) \otimes (a',b')_{\mathcal{W}}^* \\ \mathcal{K}[(a,b)_{\mathcal{W}} \otimes (a',b')_{\mathcal{W}}] &= \mathcal{W}(\Delta_{3p-a,b}) \otimes (a',b')_{\mathcal{W}} = \mathcal{W}(\Delta_{2p-a,b}) \otimes \mathcal{W}(\Delta_{3p-a',b'}) \\ &= \mathcal{W}(\Delta_{3p-a,b}) \otimes (a',b')_{\mathcal{W}}^* \\ \mathcal{L}[(a,b)_{\mathcal{W}} \otimes (a',b')_{\mathcal{W}}] &= \mathcal{W}(\Delta_{2p-a,b}) \otimes (a',b')_{\mathcal{W}} \\ \mathcal{L}^2[(a,b)_{\mathcal{W}} \otimes (a',b')_{\mathcal{W}}] &= \mathcal{W}(\Delta_{2p-a,b}) \otimes \mathcal{W}(\Delta_{2p-a',b'}) = \mathcal{W}(\Delta_{3p-a,b}) \otimes \mathcal{W}(\Delta_{3p-a',b'}) \\ &= \mathcal{W}(\Delta_{2p-a,b}) \otimes (a',b')_{\mathcal{W}}^* = (a,b)_{\mathcal{W}}^* \otimes (a',b')_{\mathcal{W}} = (a,b)_{\mathcal{W}}^* \otimes (a',b')_{\mathcal{W}}^* \\ \mathcal{M}[(a,b)_{\mathcal{W}} \otimes (a',b')_{\mathcal{W}}] &= \mathcal{W}(\Delta_{p-a,b}) \otimes (a',b')_{\mathcal{W}} \\ \mathcal{M}^2[(a,b)_{\mathcal{W}} \otimes (a',b')_{\mathcal{W}}] &= \mathcal{W}(\Delta_{p-a,b}) \otimes \mathcal{W}(\Delta_{p-a',b'}) \end{aligned} \quad (3.27)$$

These applications of $\mathcal{K}, \mathcal{L}, \mathcal{L}^2, \mathcal{M}, \mathcal{M}^2$ are trivially evaluated and their results are listed in Appendix A.2 for completeness. In particular, we find that

$$\begin{aligned} \mathcal{W}(\Delta_{a,b}) \otimes \mathcal{W}(\Delta_{a',b'}) &= \mathcal{M}^2[(p-a,b)_{\mathcal{W}} \otimes (p-a',b')_{\mathcal{W}}] \\ &= \mathcal{M}^2 \left[\bigoplus_{\substack{p-|p-a-a'|-1 \\ a''=|a-a'|+1, \text{ by } 2}} \bigoplus_{\substack{p'-|p'-b-b'|-1 \\ b''=|b-b'|+1, \text{ by } 2}} (a'',b'')_{\mathcal{W}} \oplus \mathcal{Q} \right] \\ &= \bigoplus_{\substack{p-|p-a-a'|-1 \\ a''=|a-a'|+1, \text{ by } 2}} \bigoplus_{\substack{p'-|p'-b-b'|-1 \\ b''=|b-b'|+1, \text{ by } 2}} \mathcal{W}(\Delta_{a'',b''}) \end{aligned} \quad (3.28)$$

where $\mathcal{Q} \in (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$ is given in (A.2). The irreducible modules $\mathcal{W}(\Delta_{a,b})$ are thus seen to generate a fusion subalgebra isomorphic to the fusion algebra of the usual *rational* minimal model $\mathcal{M}(p,p')$ [18, 19]. This was also observed in [7].

Proposition 1 implies the existence of a triplet of symmetry-generating modules whose fusion rules correspond to the action of the three maps \mathcal{K}, \mathcal{L} and \mathcal{M} .

Proposition 2 For $p > 1$, the \mathcal{K} -, \mathcal{L} - and \mathcal{M} -symmetries of $\text{Conj}[\mathcal{WLM}(p,p')]$ are governed by the three modules $\mathcal{W}(\Delta_{3p-1,1})$, $\mathcal{W}(\Delta_{2p-1,1})$ and $\mathcal{W}(\Delta_{p-1,1})$, respectively, in the sense that

$$\mathcal{K}[\mathcal{R}] = \mathcal{W}(\Delta_{3p-1,1}) \otimes \mathcal{R}, \quad \mathcal{L}[\mathcal{R}] = \mathcal{W}(\Delta_{2p-1,1}) \otimes \mathcal{R}, \quad \mathcal{M}[\mathcal{R}] = \mathcal{W}(\Delta_{p-1,1}) \otimes \mathcal{R} \quad (3.29)$$

¹Similar fusion rules have been conjectured independently by Simon Wood [17].

for all $\mathcal{R} \in (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$.

Proof This is an immediate consequence of

$$\mathcal{O}[\mathcal{R}] = \mathcal{O}[(1,1)_{\mathcal{W}} \otimes \mathcal{R}] = \mathcal{O}[(1,1)_{\mathcal{W}}] \otimes \mathcal{R}, \quad \mathcal{O} \in \{\mathcal{K}, \mathcal{L}, \mathcal{M}\}, \quad \mathcal{R} \in (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \quad (3.30)$$

where

$$\mathcal{K}[(1,1)_{\mathcal{W}}] = \mathcal{W}(\Delta_{3p-1,1}), \quad \mathcal{L}[(1,1)_{\mathcal{W}}] = \mathcal{W}(\Delta_{2p-1,1}), \quad \mathcal{M}[(1,1)_{\mathcal{W}}] = \mathcal{W}(\Delta_{p-1,1}) \quad (3.31)$$

□

For simple reference, we note that

$$\Delta_{kp-1,1} = \Delta_{1,kp'-1} = \frac{1}{4}(kp-2)(kp'-2) \quad (3.32)$$

In $\text{Fund}[\mathcal{WLM}(p,p')]$, the two modules $(2,1)_{\mathcal{W}}$ and $(1,2)_{\mathcal{W}}$ (in addition to $(1,1)_{\mathcal{W}}$) are naturally considered *fundamental*, cf. (2.18). In $\text{Conj}[\mathcal{WLM}(p,p')]$ for $p > 1$, on the other hand, we see that the three irreducible modules $\mathcal{W}(\Delta_{kp-1,1})$, $k \in \mathbb{Z}_{1,3}$, too, should be regarded as fundamental since

$$\text{Conj}[\mathcal{WLM}(p,p')] = \langle (1,1)_{\mathcal{W}}, (2,1)_{\mathcal{W}}, (1,2)_{\mathcal{W}}, \mathcal{W}(\Delta_{p-1,1}), \mathcal{W}(\Delta_{2p-1,1}), \mathcal{W}(\Delta_{3p-1,1}) \rangle \quad (3.33)$$

To avoid confusion, we propose to refer to the six modules appearing explicitly in (3.33) as *basic* modules. The fusion rules for these modules are summarized in Appendix A.3.

Even though the conjugation \mathcal{C} is an involution on $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$, it is *not* a symmetry of the fusion algebra $\text{Conj}[\mathcal{WLM}(p,p')]$ for $p > 1$. This is illustrated by

$$\mathcal{C}[(1,1)_{\mathcal{W}}] \otimes (1,1)_{\mathcal{W}}^* = (1,1)_{\mathcal{W}}^* \neq (1,1)_{\mathcal{W}} = \mathcal{C}[(1,1)_{\mathcal{W}} \otimes (1,1)_{\mathcal{W}}^*] \quad (3.34)$$

Instead, one verifies the following weaker result.

Proposition 3

$$\mathcal{F} \otimes \mathcal{R}^* = (\mathcal{F} \otimes \mathcal{R})^*, \quad \mathcal{F} \in (\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}, \quad \mathcal{R} \in (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \quad (3.35)$$

There are several results of the form appearing in (3.35), such as $\mathcal{W}(\Delta_{3p-a,b}) \otimes \mathcal{C}[(a',b')_{\mathcal{W}}]^* = \mathcal{C}[\mathcal{W}(\Delta_{3p-a,b}) \otimes (a',b')_{\mathcal{W}}]$, for example, but we do not exhaust here the various possible extensions of Proposition 3. We also note that $\mathcal{R}^* \otimes \mathcal{R}'^*$ is not, in general, equal to $(\mathcal{R} \otimes \mathcal{R}')^*$ as illustrated by setting $\mathcal{R} = (1,1)_{\mathcal{W}}$ and $\mathcal{R}' = (1,1)_{\mathcal{W}}^*$ for $p > 1$

$$((1,1)_{\mathcal{W}})^* \otimes ((1,1)_{\mathcal{W}}^*)^* = (1,1)_{\mathcal{W}}^* \neq (1,1)_{\mathcal{W}} = ((1,1)_{\mathcal{W}} \otimes (1,1)_{\mathcal{W}}^*)^* \quad (3.36)$$

3.4 Fusion algebra generated from irreducible modules

A simple inspection of the fusion algebra $\text{Conj}[\mathcal{WLM}(p,p')]$ reveals that the modules of the form $(a,b)_{\mathcal{W}}$ do *not* appear as the result of fusions involving a module in $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \setminus \{(a,b)_{\mathcal{W}}\}$. One also observes that all but the modules $(a,b)_{\mathcal{W}}$ are generated by repeated fusions of the irreducible modules (3.1). This implies, in particular, that the fusion algebra $\langle (\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}} \rangle$ generated from repeated fusions of the irreducible modules is a *subalgebra* of $\text{Conj}[\mathcal{WLM}(p,p')]$. For $p > 1$, this subalgebra does *not* have a unit since $(1,1)_{\mathcal{W}}$ is in the omitted set $\{(a,b)_{\mathcal{W}}\}$.

As already indicated, the spectrum of modules underlying the fusion algebra $\langle (\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}} \rangle$ is given by $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \setminus \{(a,b)_{\mathcal{W}}\}$ and is obviously not invariant under conjugation for $p > 1$. The minimal conjugation-invariant extension of this set is $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$. Thus, requiring that the spectrum is invariant under conjugation brings back the fusion algebra $\text{Conj}[\mathcal{WLM}(p,p')]$ and hence the identity $(1,1)_{\mathcal{W}}$. We recall, though, that conjugation is *not* a symmetry of this fusion algebra for $p > 1$, cf. the discussion leading up to Proposition 3.

3.5 On fusion subalgebras and quotients

So far, we have encountered

$$\langle (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \rangle \subset \langle (\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}} \rangle \subset \langle (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \rangle, \quad \langle (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \rangle \subset \langle (\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}} \rangle \subset \langle (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \rangle \quad (3.37)$$

as sequences of fusion (sub)algebras. For $p > 1$, the two sets $(\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}$ and $(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}$ are not related by \subset , while the various extensions of $(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$ are trivial for $p = 1$

$$\langle (\mathcal{J}_{1,p'}^{\text{out}})_{\mathcal{W}} \rangle = \langle (\mathcal{J}_{1,p'}^{\text{fund}})_{\mathcal{W}} \rangle = \langle (\mathcal{J}_{1,p'}^{\text{irr}})_{\mathcal{W}} \rangle = \langle (\mathcal{J}_{1,p'}^{\text{conj}})_{\mathcal{W}} \rangle \quad (3.38)$$

The fusion algebra $\text{Conj}[\mathcal{WLM}(p, p')]$ contains many other fusion subalgebras than the ones listed above. As some of these are ideals, one may also consider the corresponding quotient structures. Here, we address some of these subalgebras and quotients.

First, the fusion algebra generated by the irreducible modules $\mathcal{W}(\Delta_{a,b})$ is such an ideal. The quotient $\langle (\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \rangle / (\{\mathcal{W}(\Delta_{a,b})\})$ is equivalent to the fusion algebra constructed as in Proposition 1 if one works with $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}} \setminus \{\mathcal{W}(\Delta_{a,b})\}$ and refrains from imposing the \mathcal{M} -symmetry. As a non-trivial subalgebra, it contains the fusion algebra $\langle (\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}} \cup \{(a, b)_{\mathcal{W}}^*, \mathcal{W}(\Delta_{(\kappa+1)p-a})\} \rangle$, which, for $p > 1$, does not have a unit. Likewise, refraining from imposing the \mathcal{K} - or \mathcal{L} -symmetry, or any combination of the three symmetries, on the corresponding subset of $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$, yields a fusion subalgebra of $\text{Conj}[\mathcal{WLM}(p, p')]$. Such a subalgebra can, in general, not be described as a quotient of $\text{Conj}[\mathcal{WLM}(p, p')]$ simply because the omitted modules do not generate an ideal. For example, let us consider the situation arising when leaving out the \mathcal{L} -symmetry and omitting the modules $\mathcal{W}(\Delta_{2p-a,b})$. This yields a perfectly well-defined fusion subalgebra even though $\mathcal{L}[\mathcal{W}(\Delta_{2p-a,b})] = (a, b)_{\mathcal{W}}^*$ prevents the modules $\mathcal{W}(\Delta_{2p-a,b})$ from generating an ideal of $\text{Conj}[\mathcal{WLM}(p, p')]$.

4 Conjecture and comparison with known results

Denoting by $\text{Irr}[\mathcal{WLM}(p, p')]$ the fusion algebra generated by repeated fusions of the irreducible modules according to the Nahm-Gaberdiel-Kausch algorithm [10, 11], we conjecture that it is identical to the fusion algebra $\langle (\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}} \rangle$ discussed above.

Conjecture

$$\text{Irr}[\mathcal{WLM}(p, p')] = \langle (\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}} \rangle \quad (4.1)$$

In support of this assertion, we first note that the irreducible module $\mathcal{W}(\Delta_{a,b})$ of the \mathcal{W} -extended Virasoro algebra is, in fact, an irreducible module of the Virasoro algebra itself, that is,

$$\mathcal{W}(\Delta_{a,b}) = \mathcal{V}(\Delta_{a,b}) \quad (4.2)$$

From the Nahm-Gaberdiel-Kausch algorithm, one thus recovers the usual rational minimal-model Virasoro fusion rules (3.28).

Due to (3.38), the comparison of our proposal for $p = 1$ with the results [12, 13] obtained by application of the Nahm-Gaberdiel-Kausch algorithm was already performed in [3]. For $p > 1$, the situation is considerably more complicated and much less is known about the implications of the algorithm. Following the earlier work [16] on fusion of irreducible Virasoro modules, the fusion algebra generated by repeated fusions of the 13 irreducible modules appearing in $\mathcal{WLM}(2, 3)$ was recently worked out in [14]. The focus there was on a conjugation-invariant spectrum, and we have verified that $\text{Conj}[\mathcal{WLM}(2, 3)]$ indeed corresponds to their results. For ease of comparison, we note that

$$\mathcal{W} = (1, 1)_{\mathcal{W}}, \quad \mathcal{Q} = (1, 2)_{\mathcal{W}}, \quad \mathcal{W}^* = (1, 1)_{\mathcal{W}}^*, \quad \mathcal{Q}^* = (1, 2)_{\mathcal{W}}^* \quad (4.3)$$

in the notation of [14]. This, of course, presupposes that the notion of the *conjugate* of a module in $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ is the same as the one employed in [14]. This is easily verified. Essential aspects of the role

played by $\mathcal{W}(\Delta_{3p-1,1}) = \mathcal{W}(7)$ in $\mathcal{WLM}(2,3)$ were described in [14]. In particular, it was found that the modules in $(\mathcal{J}_{2,3}^{\text{out}})_{\mathcal{W}} \cup \{\mathcal{W}(5), \mathcal{W}(7)\} \cup \{\mathcal{W}^*, \mathcal{Q}^*\}$ are organized in pairs with respect to fusion with $\mathcal{W}(7)$. From our perspective, this corresponds to the maps

$$\mathcal{Q}_{\kappa} \xleftrightarrow{\mathcal{K}} \mathcal{Q}_{2 \cdot \kappa} \quad \mathcal{W}(\Delta_{3p-a,b}) \xleftrightarrow{\mathcal{K}} (a,b)_{\mathcal{W}}^* \quad (4.4)$$

where $\mathcal{Q}_{\kappa} \in (\mathcal{J}_{2,3}^{\text{out}})_{\mathcal{W}}$, $a \in \mathbb{Z}_{1,p-1} = \{1\}$ and $b \in \mathbb{Z}_{1,p'-1} = \{1, 2\}$.

Adopting some further terminology used in [14], though without going into details, we find that the set of modules which have a *dual* module is given by $(\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}$ and recall that they generate a closed fusion algebra, namely $\text{Fund}[\mathcal{WLM}(p,p')]$. Believing that the modules in $(\mathcal{J}_{p,p'}^{\text{fund}})_{\mathcal{W}}$ are, in fact, self-dual, it follows that the set of self-conjugate and self-dual modules is given by $(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$. This is exactly the set of modules naturally associated with \mathcal{W} -invariant boundary conditions. They, too, generate a closed fusion algebra, namely $\text{Out}[\mathcal{WLM}(p,p')]$.

5 Polynomial fusion rings

The fusion algebra

$$\phi_i \otimes \phi_j = \bigoplus_{k \in \mathcal{J}} N_{i,j}^k \phi_k, \quad i, j \in \mathcal{J} \quad (5.1)$$

of a *rational* conformal field theory is finite and can be represented by a commutative matrix algebra $\langle N_i; i \in \mathcal{J} \rangle$ where the entries of the $|\mathcal{J}| \times |\mathcal{J}|$ matrix N_i are

$$(N_i)_j^k = N_{i,j}^k, \quad i, j, k \in \mathcal{J} \quad (5.2)$$

and where the fusion multiplication \otimes has been replaced by ordinary matrix multiplication. In [20], Gepner found that every such algebra is isomorphic to a ring of polynomials in a finite set of variables modulo an ideal defined as the vanishing conditions of a finite set of polynomials in these variables. He also conjectured that this ideal of constraints corresponds to the local extrema of a potential, see [21, 22, 23] for further elaborations on this conjecture.

We extended Gepner's result to the fundamental fusion algebra $\text{Fund}[\mathcal{WLM}(p,p')]$ in [6] where we found that

$$\text{Fund}[\mathcal{WLM}(p,p')] \simeq \mathbb{C}[X, Y] / (P_p(X), P_{p'}(Y), P_{p,p'}(X, Y)) \quad (5.3)$$

Here,

$$P_n(x) = 2\left(T_{2n}\left(\frac{x}{2}\right) - 1\right)U_{n-1}\left(\frac{x}{2}\right), \quad P_{n,n'}(x, y) = \left(T_n\left(\frac{x}{2}\right) - T_{n'}\left(\frac{y}{2}\right)\right)U_{n-1}\left(\frac{x}{2}\right)U_{n'-1}\left(\frac{y}{2}\right) \quad (5.4)$$

where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively. The isomorphism in (5.3) reads

$$\begin{aligned} (a, b)_{\mathcal{W}} &\leftrightarrow U_{a-1}\left(\frac{X}{2}\right)U_{b-1}\left(\frac{Y}{2}\right) \\ \mathcal{W}(\Delta_{\kappa p, s}) &\leftrightarrow \frac{1}{\kappa}U_{\kappa p-1}\left(\frac{X}{2}\right)U_{s-1}\left(\frac{Y}{2}\right) \\ \mathcal{W}(\Delta_{a, \kappa p'}) &\leftrightarrow \frac{1}{\kappa}U_{a-1}\left(\frac{X}{2}\right)U_{\kappa p'-1}\left(\frac{Y}{2}\right) \\ (\mathcal{R}_{\kappa p, s}^{a, 0})_{\mathcal{W}} &\leftrightarrow \frac{2}{\kappa}T_a\left(\frac{X}{2}\right)U_{\kappa p-1}\left(\frac{X}{2}\right)U_{s-1}\left(\frac{Y}{2}\right) \\ (\mathcal{R}_{r, \kappa p'}^{0, b})_{\mathcal{W}} &\leftrightarrow \frac{2}{\kappa}U_{r-1}\left(\frac{X}{2}\right)T_b\left(\frac{Y}{2}\right)U_{\kappa p'-1}\left(\frac{Y}{2}\right) \\ (\mathcal{R}_{\kappa p, p'}^{a, b})_{\mathcal{W}} &\leftrightarrow \frac{4}{\kappa}T_a\left(\frac{X}{2}\right)U_{\kappa p-1}\left(\frac{X}{2}\right)T_b\left(\frac{Y}{2}\right)U_{p'-1}\left(\frac{Y}{2}\right) \end{aligned} \quad (5.5)$$

where it is noted that

$$U_{\kappa p-1}\left(\frac{X}{2}\right)U_{p'-1}\left(\frac{Y}{2}\right) \equiv U_{p-1}\left(\frac{X}{2}\right)U_{\kappa p'-1}\left(\frac{Y}{2}\right) \pmod{P_{p,p'}(X,Y)} \quad (5.6)$$

for example.

We now wish to show that the fusion algebra $\text{Conj}[\mathcal{WLM}(p,p')]$ also admits a polynomial-ring description. For $p = 1$, this is trivially true since $\text{Conj}[\mathcal{WLM}(1,p')] = \text{Fund}[\mathcal{WLM}(1,p')]$. In the following, we will therefore assume that $p > 1$.

According to Proposition 1, the fusion algebra $\text{Conj}[\mathcal{WLM}(p,p')]$ is constructed as an extension of $\text{Fund}[\mathcal{WLM}(p,p')]$ where the extension, according to Proposition 2, is governed by the three irreducible modules $\mathcal{W}(\Delta_{kp-1,1})$, $k \in \mathbb{Z}_{1,3}$. Following from the diagrams (3.20), (3.21) and the ensuing composition algebra in Figure 1, we find that $\text{Conj}[\mathcal{WLM}(p,p')]$ can be described by a quotient polynomial ring with five generators, cf. (3.33).

Proposition 4 For $p > 1$, the fusion algebra $\text{Conj}[\mathcal{WLM}(p,p')]$ is isomorphic to the quotient polynomial ring

$$\text{Conj}[\mathcal{WLM}(p,p')] \simeq \mathbb{C}[X, Y, K, L, M] / \mathcal{I}_{p,p'} \quad (5.7)$$

where $\mathcal{I}_{p,p'}$ is the ideal defined by the vanishing conditions

$$\begin{aligned} 0 &= \left(K - T_p\left(\frac{X}{2}\right)\right)U_{p-1}\left(\frac{X}{2}\right) = \left(K - T_{p'}\left(\frac{Y}{2}\right)\right)U_{p'-1}\left(\frac{Y}{2}\right) = (L-1)U_{p-1}\left(\frac{X}{2}\right) = (L-1)U_{p'-1}\left(\frac{Y}{2}\right) \\ &= M^2 - MU_{p-2}\left(\frac{X}{2}\right) = M^2 - MU_{p'-2}\left(\frac{Y}{2}\right) = K(L-1) = K^2 - L^2 = KM = LM \end{aligned} \quad (5.8)$$

The isomorphism in (5.7) is given by (5.5) supplemented by

$$\begin{aligned} \mathcal{W}(\Delta_{p-a,b}) &\leftrightarrow MU_{a-1}\left(\frac{X}{2}\right)U_{b-1}\left(\frac{Y}{2}\right) \\ \mathcal{W}(\Delta_{2p-a,b}) &\leftrightarrow LU_{a-1}\left(\frac{X}{2}\right)U_{b-1}\left(\frac{Y}{2}\right) \\ \mathcal{W}(\Delta_{3p-a,b}) &\leftrightarrow KU_{a-1}\left(\frac{X}{2}\right)U_{b-1}\left(\frac{Y}{2}\right) \\ (a,b)_{\mathcal{W}}^* &\leftrightarrow L^2U_{a-1}\left(\frac{X}{2}\right)U_{b-1}\left(\frac{Y}{2}\right) \end{aligned} \quad (5.9)$$

Proof With

$$K \leftrightarrow \mathcal{W}(\Delta_{3p-1,1}), \quad L \leftrightarrow \mathcal{W}(\Delta_{2p-1,1}), \quad M \leftrightarrow \mathcal{W}(\Delta_{p-1,1}) \quad (5.10)$$

it follows that $\text{Conj}[\mathcal{WLM}(p,p')]$ is isomorphic to the quotient polynomial ring in X, Y, K, L and M whose defining ideal can be described by supplementing the conditions in (5.3) with the conditions following from translating the various arrows (including the trivial identity maps) in the diagrams (3.20) and (3.21) into polynomial constraints. Completing the proof thus amounts to verifying that the set of conditions in (5.8) is *necessary* and *sufficient* to characterize this ideal. This is straightforwardly done. Here, we only include a couple of these verifications as the remaining ones are treated similarly. First, that a condition is necessary means that it is a consequence of the conditions given in (5.3) combined with the ones following from the translation procedure. From $\mathcal{K}[\mathcal{W}(\Delta_{p,1})] = \mathcal{W}(\Delta_{2p,1})$, for example, we thus conclude that $KU_{p-1}\left(\frac{X}{2}\right) = \frac{1}{2}U_{2p-1}\left(\frac{X}{2}\right)$. Using the identity $U_{2p-1}(x) = 2T_p(x)U_{p-1}(x)$, we immediately recognize the first condition appearing in (5.8). To illustrate that the conditions in (5.8) are sufficient, we observe that

$$MU_{\kappa p-1}\left(\frac{X}{2}\right) \equiv 0 \pmod{(L-1)U_{p-1}\left(\frac{X}{2}\right), LM} \quad (5.11)$$

where the congruence for $\kappa = 2$ is a simple consequence of the one for $\kappa = 1$. Multiplied by $U_{s-1}(\frac{Y}{2})$, this corresponds to $\mathcal{M}[\mathcal{W}(\Delta_{\kappa p, s})] = 0$. Using the identity

$$U_{p-2}(x)U_{a-1}(x) = U_{p-a-1}(x) + \sum_{n=-(a-2), \text{ by } 2}^{a-2} T_{|n|}(x)U_{p-1}(x) \quad (5.12)$$

we subsequently find that

$$M^2 U_{a-1}(\frac{X}{2}) \equiv M U_{p-a-1}(\frac{X}{2}), \quad (\text{mod } M^2 - M U_{p-2}(\frac{X}{2}), M U_{p-1}(\frac{X}{2})) \quad (5.13)$$

which, multiplied by $U_{b-1}(\frac{Y}{2})$, corresponds to $\mathcal{M}^2[(a, b)_{\mathcal{W}}] = \mathcal{M}[(p-a, b)_{\mathcal{W}}]$ which itself comes from $\mathcal{M}[\mathcal{W}(\Delta_{p-a, b})] = \mathcal{W}(\Delta_{a, b})$. Let us also consider $\mathcal{K}[(\mathcal{R}_{\kappa p, p'}^{a, b})_{\mathcal{W}}] = [(\mathcal{R}_{(2 \cdot \kappa)p, p'}^{a, b})_{\mathcal{W}}]$ corresponding to

$$K \frac{4}{\kappa} T_a(\frac{X}{2}) U_{\kappa p-1}(\frac{X}{2}) T_b(\frac{Y}{2}) U_{p'-1}(\frac{Y}{2}) \equiv \frac{4}{2 \cdot \kappa} T_a(\frac{X}{2}) U_{(2 \cdot \kappa)p-1}(\frac{X}{2}) T_b(\frac{Y}{2}) U_{p'-1}(\frac{Y}{2}) \quad (5.14)$$

For $\kappa = 1$, this follows immediately from $0 = (K - T_p(\frac{X}{2}))U_{p-1}(\frac{X}{2})$. For $\kappa = 2$, it follows from $P_p(X) = 0$ which is not, though, in the set (5.8). However, we wish to emphasize that, not only $P_p(X) = 0$, but all three conditions appearing in (5.3) are consequences of the conditions in (5.8). This follows from

$$P_{p, p'}(X, Y) = \left(K - T_{p'}(\frac{Y}{2})\right) U_{p-1}(\frac{X}{2}) U_{p'-1}(\frac{Y}{2}) - \left(K - T_p(\frac{X}{2})\right) U_{p-1}(\frac{X}{2}) U_{p'-1}(\frac{Y}{2}) \equiv 0 \quad (5.15)$$

where the congruence is modulo $(K - T_p(\frac{X}{2}))U_{p-1}(\frac{X}{2})$ and $(K - T_{p'}(\frac{Y}{2}))U_{p'-1}(\frac{Y}{2})$, and from

$$P_p(X) = 4 \left(T_p^2(\frac{X}{2}) - 1\right) U_{p-1}(\frac{X}{2}) \equiv 0 \quad (5.16)$$

where the congruence is modulo $(L - 1)U_{p-1}(\frac{X}{2})$, $K^2 - L^2$ and $(K - T_p(\frac{X}{2}))U_{p-1}(\frac{X}{2})$. The condition for $P_{p'}(Y)$ follows similarly, of course.

□

Just as $P_p(X)$ and $P_{p'}(Y)$ are the minimal polynomials of X and Y modulo $\mathcal{I}_{p, p'}$, we see that $K(K^2 - 1)$, $L^2(L - 1)$ and $M(M^2 - 1)$ are the minimal polynomials of K , L and M , respectively. Indeed, using (5.13), in particular, we have

$$K^3 \equiv K L^2 \equiv K L \equiv K, \quad L^3 \equiv K^2 L \equiv K^2 \equiv L^2, \quad M^3 \equiv M^2 U_{p-2}(\frac{X}{2}) \equiv M \quad (5.17)$$

modulo $\mathcal{I}_{p, p'}$.

From the analysis above, we extract the conditions linking the modules $\mathcal{W}(\Delta_{a, b})$ to each other

$$0 = M U_{p-1}(\frac{X}{2}) = M U_{p'-1}(\frac{Y}{2}) = M \left(U_{p-2}(\frac{X}{2}) - U_{p'-2}(\frac{Y}{2}) \right) \quad (5.18)$$

Up to the factors of M , these are recognized as the standard conditions defining the quotient polynomial ring associated with the *rational* minimal models, see [19, 6], for example. This should not, though, come as a surprise since we have already realized that the fusion subalgebra generated by the irreducible modules $\mathcal{W}(\Delta_{a, b})$ satisfy the usual minimal-model fusion rules (3.28).

Since the minimal fusion algebra generated from repeated fusions of the irreducible modules, $\langle (\mathcal{J}_{p, p'}^{\text{irr}})_{\mathcal{W}} \rangle$, does not have a unit, it cannot be isomorphic to a quotient polynomial ring. However, since this fusion algebra is a subalgebra of $\text{Conj}[\mathcal{WLM}(p, p')] = \langle (\mathcal{J}_{p, p'}^{\text{conj}})_{\mathcal{W}} \rangle$, it is isomorphic to a *subring* of the quotient polynomial ring appearing in (5.7). This subring is obtained by omitting the

polynomials $U_{a-1}(\frac{X}{2})U_{b-1}(\frac{Y}{2})$ themselves from the ambient ring while keeping their products with other non-trivial polynomials. We see that this corresponds to eliminating the identity map \mathcal{I} from the allowed operations on $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ when constructing the composition algebra in Figure 1. This elimination procedure is algebraically well-defined, cf. the closure of the composition algebra and the discussion following (3.19). We also recall from [6] that omitting the polynomials $U_{a-1}(\frac{X}{2})U_{b-1}(\frac{Y}{2})$ from the quotient polynomial ring in (5.3) yields a well-defined subring isomorphic to the fusion algebra $\text{Out}[\mathcal{WLM}(p, p')]$ of the modules naturally associated with \mathcal{W} -invariant boundary conditions.

6 Concluding remarks

Based on symmetry principles, we have derived a fusion algebra $\langle(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}\rangle$ generated from repeated fusions of the irreducible modules appearing in the \mathcal{W} -extended logarithmic minimal model $\mathcal{WLM}(p, p')$. In addition to the irreducible modules themselves (3.1), closure of the fusion algebra requires the participation of a variety of reducible yet indecomposable modules. We conjecture that this fusion algebra is the same as the fusion algebra $\text{Irr}[\mathcal{WLM}(p, p')]$, also generated by repeated fusions of the irreducible modules, but obtained by application of the Nahm-Gaberdiel-Kausch algorithm. In support of this conjecture, we find that the two fusion algebras agree for $\mathcal{WLM}(1, p')$ [12, 13, 3] and for $\mathcal{WLM}(2, 3)$ [14]. For $p > 1$, our fusion algebra does not contain an algebra unit. Requiring that the spectrum of modules is invariant under a natural notion of conjugation, however, introduces an additional $(p-1)(p'-1)$ reducible yet indecomposable rank-1 modules, among which the identity is found. This bigger set of indecomposable modules is denoted by $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$. The corresponding fusion algebra $\text{Conj}[\mathcal{WLM}(p, p')]$ is invariant under the symmetries \mathcal{K} , \mathcal{L} and \mathcal{M} and contains $\text{Irr}[\mathcal{WLM}(p, p')]$ (or strictly speaking $\langle(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}\rangle$) as a subalgebra. These symmetry generators are maps from $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ to itself. Their actions on $\text{Conj}[\mathcal{WLM}(p, p')]$ are shown to be generated by fusions with the three irreducible modules of conformal weights $\Delta_{kp-1,1}$, $k = 3, 2, 1$, respectively. We have also identified a polynomial fusion ring isomorphic to the fusion algebra $\text{Conj}[\mathcal{WLM}(p, p')]$. For $p > 1$, it has five generators corresponding to the two fundamental modules and the three symmetry generators. The fusion algebra $\langle(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}\rangle$ is isomorphic to a particular subring of the polynomial fusion ring.

The reader may wonder about our motivation for introducing the three maps \mathcal{K} , \mathcal{L} and \mathcal{M} when the objective was to determine the fusion algebra generated by repeated fusions of the irreducible modules. First, since $\mathcal{W}(\Delta_{a,b}) = \mathcal{V}(\Delta_{a,b})$, we want the irreducible modules $\mathcal{W}(\Delta_{a,b})$ to generate a fusion subalgebra isomorphic to the fusion algebra of the Virasoro minimal model $\mathcal{M}(p, p')$. Second, from the success of $\text{Fund}[\mathcal{WLM}(p, p')]$ as an ambient fusion algebra hosting the fusion algebra $\text{Out}[\mathcal{WLM}(p, p')]$ generated by the modules associated with boundary conditions, we expect to encounter a fusion algebra generated by a small number of basic modules. As in the case of $\text{Fund}[\mathcal{WLM}(p, p')]$, this fusion algebra may not be the sought-after fusion algebra itself ($\text{Irr}[\mathcal{WLM}(p, p')]$), but rather an extension thereof ($\text{Conj}[\mathcal{WLM}(p, p')]$). Third, examinations like (3.8) reveal that the contragredient modules $(a, b)_{\mathcal{W}}^*$ are generated, and we are led to consider the set $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ of indecomposable modules. Fourth, we wish to preserve as much as possible the factorization enjoyed by the indecomposable modules of the fundamental fusion algebra where every module can be written as the fusion of a “horizontal” and a “vertical” module: $(\mathcal{R}_{\rho,\sigma}^{\alpha,\beta})_{\mathcal{W}} = (\mathcal{R}_{\rho,1}^{\alpha,0})_{\mathcal{W}} \otimes (\mathcal{R}_{1,\sigma}^{0,\beta})_{\mathcal{W}}$. Supported by explicit evaluations, we then made the ansatz that the set of basic modules is given by the ones appearing explicitly in (3.33), that is,

$$\{(1, 1)_{\mathcal{W}}, (2, 1)_{\mathcal{W}}, (1, 2)_{\mathcal{W}}, \mathcal{W}(\Delta_{p-1,1}), \mathcal{W}(\Delta_{2p-1,1}), \mathcal{W}(\Delta_{3p-1,1})\} \quad (6.1)$$

and that their fusion rules are the ones given in Figure 3 and 4. For $p > 2$, we thus have

$$\begin{aligned} (1, 2)_{\mathcal{W}} \otimes \mathcal{W}(\Delta_{kp-1,1}) &= \mathcal{W}(\Delta_{kp-1,2}) \\ (2, 1)_{\mathcal{W}} \otimes \mathcal{W}(\Delta_{1, kp'-1}) &= (2, 1)_{\mathcal{W}} \otimes \mathcal{W}(\Delta_{kp-1,1}) = \mathcal{W}(\Delta_{kp-2,1}) = \mathcal{W}(\Delta_{2, kp'-1}) \end{aligned} \quad (6.2)$$

for example, resembling the aforementioned factorization. Everything else is fixed by requiring associativity of the fusion algebra, the one called $\text{Conj}[\mathcal{WLM}(p, p')]$. The subalgebra generated from repeated

fusions of the irreducible modules is subsequently identified straightforwardly. Since universality, as opposed to model-specific properties, is likely to be manifest when the basic rules of the game are expressed in terms of symmetry principles, we found it natural to try to translate the fusion rules into such principles thereby introducing \mathcal{K} , \mathcal{L} and \mathcal{M} . Once identified, these symmetry generators illuminate quite clearly the structure of the fusion algebra.

In summary, we have verified that our proposals provide well-defined fusion algebras ($\langle(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}\rangle$ and $\text{Conj}[\mathcal{WLM}(p,p')]$, in particular) and that they reproduce all known results in this regard. We are not, though, making any claims of uniqueness of the constructions, but do conjecture that $\langle(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}\rangle$ is identical to $\text{Irr}[\mathcal{WLM}(p,p')]$ obtained by application of the Nahm-Gaberdiel-Kausch algorithm. As generated from the minimal conjugation-invariant extension $(\mathcal{J}_{p,p'}^{\text{conj}})_{\mathcal{W}}$ of the spectrum $(\mathcal{J}_{p,p'}^{\text{irr}})_{\mathcal{W}}$, we furthermore conjecture that the \mathcal{K} -, \mathcal{L} - and \mathcal{M} -invariant fusion algebra $\text{Conj}[\mathcal{WLM}(p,p')]$ is identical to the similar extension obtained by application of the Nahm-Gaberdiel-Kausch algorithm.

Note added

After the present work appeared on the arXiv, the paper [17] appeared on the arXiv. As a continuation of the work [14], it also addresses the fusion algebra generated from repeated fusions of irreducible modules, and it presents conjectured fusion rules similar to the ones proposed in Section 3.3.

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A Fusion rules of $\text{Conj}[\mathcal{WLM}(p,p')]$

A.1 Fundamental fusion algebra $\text{Fund}[\mathcal{WLM}(p,p')]$

Here, we summarize the fusion rules, obtained in [5, 6], underlying the fusion algebra $\text{Fund}[\mathcal{WLM}(p,p')]$ as given in (2.17). To this end, by a direct sum of representations A_n with unspecified lower summation bound, we mean the direct sum in steps of 2 whose lower bound is given by the parity of the upper bound

$$\bigoplus_n^N A_n = \bigoplus_{n=\frac{1}{2}(1-(-1)^N), \text{ by } 2}^N A_n, \quad N \in \mathbb{Z} \quad (\text{A.1})$$

This direct sum vanishes for negative N . For simplicity, and in compliance with the notation of [6], we write $(\mathcal{R}_{\rho,\sigma}^{0,0})_{\mathcal{W}} = (\rho,\sigma)_{\mathcal{W}}$, $(\kappa p,s)_{\mathcal{W}} = \mathcal{W}(\Delta_{\kappa p,s})$ and $(r,\kappa p')_{\mathcal{W}} = \mathcal{W}(\Delta_{r,\kappa p'})$. Now, the fusions involving the module $(a,b)_{\mathcal{W}}$ are given by

$$\begin{aligned} (a,b)_{\mathcal{W}} \otimes (a',b')_{\mathcal{W}} = & \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \left\{ \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (i,j)_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{a+a'-p-1} \left\{ \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (\mathcal{R}_{p,j}^{\alpha,0})_{\mathcal{W}} \right\} \\ & \oplus \bigoplus_{\beta}^{b+b'-p'-1} \left\{ \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} (\mathcal{R}_{i,p'}^{0,\beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{a+a'-p-1} \left\{ \bigoplus_{\beta}^{b+b'-p'-1} (\mathcal{R}_{p,p'}^{\alpha,\beta})_{\mathcal{W}} \right\} \quad (\text{A.2}) \end{aligned}$$

and

$$\begin{aligned}
(a, b)_{\mathcal{W}} \otimes (\kappa p, b')_{\mathcal{W}} &= \bigoplus_{\alpha}^{a-1} \left\{ \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (\mathcal{R}_{\kappa p, j}^{\alpha, 0})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b+b'-p'-1} (\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(a, b)_{\mathcal{W}} \otimes (a', \kappa p')_{\mathcal{W}} &= \bigoplus_{\beta}^{b-1} \left\{ \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} (\mathcal{R}_{i, \kappa p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+a'-p-1} (\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(a, b)_{\mathcal{W}} \otimes (\kappa p, p')_{\mathcal{W}} &= \bigoplus_{\alpha}^{a-1} \left\{ \bigoplus_{\beta}^{b-1} (\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \tag{A.3}
\end{aligned}$$

and

$$\begin{aligned}
(a, b)_{\mathcal{W}} \otimes (\mathcal{R}_{\kappa p, s}^{a', 0})_{\mathcal{W}} &= \bigoplus_{\alpha=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \left\{ \bigoplus_{j=|b-s|+1, \text{ by } 2}^{p'-|p'-b-s|-1} (\mathcal{R}_{\kappa p, j}^{\alpha, 0})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b+s-p'-1} (\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a-a'-1} \left\{ \bigoplus_{j=|b-s|+1, \text{ by } 2}^{p'-|p'-b-s|-1} 2(\mathcal{R}_{\kappa p, j}^{\alpha, 0})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b+s-p'-1} 2(\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a+a'-p-1} \left\{ \bigoplus_{j=|b-s|+1, \text{ by } 2}^{p'-|p'-b-s|-1} 2(\mathcal{R}_{(2 \cdot \kappa)p, j}^{\alpha, 0})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b+s-p'-1} 2(\mathcal{R}_{(2 \cdot \kappa)p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(a, b)_{\mathcal{W}} \otimes (\mathcal{R}_{r, \kappa p'}^{0, b'})_{\mathcal{W}} &= \bigoplus_{\beta=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} \left\{ \bigoplus_{i=|a-r|+1, \text{ by } 2}^{p-|p-a-r|-1} (\mathcal{R}_{i, \kappa p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+r-p-1} (\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b-b'-1} \left\{ \bigoplus_{i=|a-r|+1, \text{ by } 2}^{p-|p-a-r|-1} 2(\mathcal{R}_{i, \kappa p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+r-p-1} 2(\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b+b'-p'-1} \left\{ \bigoplus_{i=|a-r|+1, \text{ by } 2}^{p-|p-a-r|-1} 2(\mathcal{R}_{i, (2 \cdot \kappa)p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+r-p-1} 2(\mathcal{R}_{(2 \cdot \kappa)p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \tag{A.4}
\end{aligned}$$

and

$$\begin{aligned}
(a, b)_{\mathcal{W}} \otimes (\mathcal{R}_{\kappa p, p'}^{a', b'})_{\mathcal{W}} &= \bigoplus_{\alpha=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \left\{ \bigoplus_{\beta=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b-b'-1} \left\{ \bigoplus_{\alpha=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} 2(\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{a-a'-1} \left\{ \bigoplus_{\beta=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} 2(\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a-a'-1} \left\{ \bigoplus_{\beta}^{b-b'-1} 4(\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{a+a'-p-1} \left\{ \bigoplus_{\beta}^{b+b'-p'-1} 4(\mathcal{R}_{\kappa p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a+a'-p-1} \left\{ \bigoplus_{\beta=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} 2(\mathcal{R}_{(2 \cdot \kappa)p, p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b-b'-1} 4(\mathcal{R}_{(2 \cdot \kappa)p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b+b'-p'-1} \left\{ \bigoplus_{\alpha=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} 2(\mathcal{R}_{(2 \cdot \kappa)p, p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a-a'-1} 4(\mathcal{R}_{(2 \cdot \kappa)p, p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \tag{A.5}
\end{aligned}$$

The fusion of two \mathcal{W} -indecomposable rank-1 modules in $(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$ is given by

$$\begin{aligned}
(\kappa p, s)_{\mathcal{W}} \otimes (\kappa' p, s')_{\mathcal{W}} &= \bigoplus_{\alpha}^{p-1} \left\{ \bigoplus_{j=|s-s'|+1, \text{ by } 2}^{p'-|p'-s-s'|-1} (\mathcal{R}_{(\kappa \cdot \kappa') p, j}^{\alpha, 0})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{s+s'-p'-1} (\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(\kappa p, s)_{\mathcal{W}} \otimes (r, \kappa' p')_{\mathcal{W}} &= \bigoplus_{\alpha}^{r-1} \left\{ \bigoplus_{\beta}^{s-1} (\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(r, \kappa p')_{\mathcal{W}} \otimes (r', \kappa' p')_{\mathcal{W}} &= \bigoplus_{\beta}^{p'-1} \left\{ \bigoplus_{j=|r-r'|+1, \text{ by } 2}^{p-|p-r-r'|-1} (\mathcal{R}_{j, (\kappa \cdot \kappa') p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{r+r'-p-1} (\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \quad (\text{A.6})
\end{aligned}$$

The fusion of a \mathcal{W} -indecomposable rank-1 module in $(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$ with a \mathcal{W} -indecomposable rank-2 module is given by

$$\begin{aligned}
(\kappa p, s)_{\mathcal{W}} \otimes (\mathcal{R}_{\kappa' p, s'}^{a, 0})_{\mathcal{W}} &= \bigoplus_{j=|s-s'|+1, \text{ by } 2}^{p'-|p'-s-s'|-1} \left\{ \bigoplus_{\alpha}^{p-a-1} 2(\mathcal{R}_{(\kappa \cdot \kappa') p, j}^{\alpha, 0})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a-1} 2(\mathcal{R}_{(2 \cdot \kappa \cdot \kappa') p, j}^{\alpha, 0})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{s+s'-p'-1} \left\{ \bigoplus_{\alpha}^{p-a-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(\kappa p, s)_{\mathcal{W}} \otimes (\mathcal{R}_{r, \kappa' p'}^{0, b})_{\mathcal{W}} &= \bigoplus_{\alpha}^{r-1} \left\{ \bigoplus_{\beta=|b-s|+1, \text{ by } 2}^{p'-|p'-s-b|-1} (\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{s-b-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b+s-p'-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(r, \kappa p')_{\mathcal{W}} \otimes (\mathcal{R}_{\kappa' p, s}^{a, 0})_{\mathcal{W}} &= \bigoplus_{\beta}^{s-1} \left\{ \bigoplus_{\alpha=|a-r|+1, \text{ by } 2}^{p-|p-r-a|-1} (\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{r-a-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+r-p-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(r, \kappa p')_{\mathcal{W}} \otimes (\mathcal{R}_{r', \kappa' p'}^{0, b})_{\mathcal{W}} &= \bigoplus_{j=|r-r'|+1, \text{ by } 2}^{p-|p-r-r'|-1} \left\{ \bigoplus_{\beta}^{p'-b-1} 2(\mathcal{R}_{j, (\kappa \cdot \kappa') p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b-1} 2(\mathcal{R}_{j, (2 \cdot \kappa \cdot \kappa') p'}^{0, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{r+r'-p-1} \left\{ \bigoplus_{\beta}^{p'-b-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \quad (\text{A.7})
\end{aligned}$$

The fusion of a \mathcal{W} -indecomposable rank-1 module in $(\mathcal{J}_{p,p'}^{\text{out}})_{\mathcal{W}}$ with a \mathcal{W} -indecomposable rank-3 module is given by

$$\begin{aligned}
(\kappa p, s)_{\mathcal{W}} \otimes (\mathcal{R}_{p, \kappa' p'}^{a, b})_{\mathcal{W}} &= \bigoplus_{\alpha}^{p-a-1} \left\{ \bigoplus_{\beta=|b-s|+1, \text{ by } 2}^{p'-|p'-s-b|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{s-b-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b+s-p'-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a-1} \left\{ \bigoplus_{\beta=|b-s|+1, \text{ by } 2}^{p'-|p'-s-b|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{s-b-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{b+s-p'-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(r, \kappa p')_{\mathcal{W}} \otimes (\mathcal{R}_{p, \kappa' p'}^{a, b})_{\mathcal{W}} &= \bigoplus_{\beta}^{p'-b-1} \left\{ \bigoplus_{\alpha=|a-r|+1, \text{ by } 2}^{p-|p-r-a|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{r-a-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+r-p-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b-1} \left\{ \bigoplus_{\alpha=|a-r|+1, \text{ by } 2}^{p-|p-r-a|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{r-a-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+r-p-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \quad (\text{A.8})
\end{aligned}$$

The fusion of two \mathcal{W} -indecomposable rank-2 modules is given by

$$\begin{aligned}
(\mathcal{R}_{\kappa p, s}^{a,0})_{\mathcal{W}} \otimes (\mathcal{R}_{\kappa' p, s'}^{a',0})_{\mathcal{W}} &= \bigoplus_{j=|s-s'|+1, \text{ by } 2}^{p'-|p'-s-s'|-1} \left\{ \bigoplus_{\alpha}^{p-|a-a'|-1} 2(\mathcal{R}_{(\kappa \cdot \kappa') p, j}^{\alpha,0})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|p-a-a'|-1} 2(\mathcal{R}_{(\kappa \cdot \kappa') p, j}^{\alpha,0})_{\mathcal{W}} \right. \\
&\quad \left. \oplus \bigoplus_{\alpha}^{p-|p-a-a'|-1} 2(\mathcal{R}_{(2 \cdot \kappa \cdot \kappa') p, j}^{\alpha,0})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|a-a'|-1} 2(\mathcal{R}_{(2 \cdot \kappa \cdot \kappa') p, j}^{\alpha,0})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{s+s'-p'-1} \left\{ \bigoplus_{\alpha}^{p-|a-a'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|p-a-a'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right. \\
&\quad \left. \oplus \bigoplus_{\alpha}^{p-|p-a-a'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|a-a'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(\mathcal{R}_{\kappa p, s}^{a,0})_{\mathcal{W}} \otimes (\mathcal{R}_{r, \kappa' p'}^{0,b})_{\mathcal{W}} &= \bigoplus_{\alpha=|a-r|+1, \text{ by } 2}^{p-|p-r-a|-1} \left\{ \bigoplus_{\beta=|b-s|+1, \text{ by } 2}^{p'-|p'-s-b|-1} (\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha=|a-r|+1, \text{ by } 2}^{p-|p-r-a|-1} \left\{ \bigoplus_{\beta}^{s-b-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\beta=|b-s|+1, \text{ by } 2}^{p'-|p'-s-b|-1} \left\{ \bigoplus_{\alpha}^{r-a-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{r-a-1} \left\{ \bigoplus_{\beta}^{s-b-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{a+r-p-1} \left\{ \bigoplus_{\beta}^{b+s-p'-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a+r-p-1} \left\{ \bigoplus_{\beta=|b-s|+1, \text{ by } 2}^{p'-|p'-s-b|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{s-b-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b+s-p'-1} \left\{ \bigoplus_{\alpha=|a-r|+1, \text{ by } 2}^{p-|p-r-a|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{r-a-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(\mathcal{R}_{r, \kappa p'}^{0,b})_{\mathcal{W}} \otimes (\mathcal{R}_{r', \kappa' p'}^{0,b'})_{\mathcal{W}} &= \bigoplus_{j=|r-r'|+1, \text{ by } 2}^{p-|p-r-r'|-1} \left\{ \bigoplus_{\beta}^{p'-|b-b'|-1} 2(\mathcal{R}_{j, (\kappa \cdot \kappa') p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|p'-b-b'|-1} 2(\mathcal{R}_{j, (\kappa \cdot \kappa') p'}^{0, \beta})_{\mathcal{W}} \right. \\
&\quad \left. \oplus \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 2(\mathcal{R}_{j, (2 \cdot \kappa \cdot \kappa') p'}^{0, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|b-b'|-1} 2(\mathcal{R}_{j, (2 \cdot \kappa \cdot \kappa') p'}^{0, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{r+r'-p-1} \left\{ \bigoplus_{\beta}^{p'-|b-b'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|p'-b-b'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right. \\
&\quad \left. \oplus \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|b-b'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \tag{A.9}
\end{aligned}$$

The fusion of a \mathcal{W} -indecomposable rank-2 module with a \mathcal{W} -indecomposable rank-3 module is given by

$$\begin{aligned}
(\mathcal{R}_{\kappa p, s}^{a, 0})_{\mathcal{W}} \otimes (\mathcal{R}_{p, \kappa' p'}^{a', b'})_{\mathcal{W}} &= \bigoplus_{\beta=|b'-s|+1, \text{ by } 2}^{p'-|p'-s-b'|-1} \left\{ \bigoplus_{\alpha}^{p-|a-a'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|p-a-a'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{s-b'-1} \left\{ \bigoplus_{\alpha}^{p-|a-a'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|p-a-a'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b'+s-p'-1} \left\{ \bigoplus_{\alpha}^{p-|p-a-a'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|a-a'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta=|b'-s|+1, \text{ by } 2}^{p'-|p'-s-b'|-1} \left\{ \bigoplus_{\alpha}^{p-|p-a-a'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|a-a'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{s-b'-1} \left\{ \bigoplus_{\alpha}^{p-|p-a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\beta}^{b'+s-p'-1} \left\{ \bigoplus_{\alpha}^{p-|a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|p-a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
(\mathcal{R}_{r, \kappa p'}^{0, b})_{\mathcal{W}} \otimes (\mathcal{R}_{p, \kappa' p'}^{a', b'})_{\mathcal{W}} &= \bigoplus_{\alpha=|a'-r|+1, \text{ by } 2}^{p-|p-r-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|b-b'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|p'-b-b'|-1} 2(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{r-a'-1} \left\{ \bigoplus_{\beta}^{p'-|b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a'+r-p-1} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha=|a'-r|+1, \text{ by } 2}^{p-|p-r-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|b-b'|-1} 2(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{r-a'-1} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
&\oplus \bigoplus_{\alpha}^{a'+r-p-1} \left\{ \bigoplus_{\beta}^{p'-|b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \quad (\text{A.10})
\end{aligned}$$

Finally, the fusion of two \mathcal{W} -indecomposable rank-3 modules is given by

$$\begin{aligned}
(\mathcal{R}_{\kappa p, p'}^{a, b})_{\mathcal{W}} \otimes (\mathcal{R}_{p, \kappa' p'}^{a', b'})_{\mathcal{W}} = & \bigoplus_{\alpha}^{p-|a-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{|p-a-a'|-1} \left\{ \bigoplus_{\beta}^{|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
\oplus & \bigoplus_{\alpha}^{p-|a-a'|-1} \left\{ \bigoplus_{\beta}^{|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{|p-a-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
\oplus & \bigoplus_{\alpha}^{p-|p-a-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{|a-a'|-1} \left\{ \bigoplus_{\beta}^{|b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
\oplus & \bigoplus_{\alpha}^{p-|p-a-a'|-1} \left\{ \bigoplus_{\beta}^{|b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \oplus \bigoplus_{\alpha}^{|a-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, \kappa' p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
\oplus & \bigoplus_{\alpha}^{p-|a-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
\oplus & \bigoplus_{\alpha}^{|p-a-a'|-1} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\beta}^{|b-b'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
\oplus & \bigoplus_{\beta}^{p'-|b-b'|-1} \left\{ \bigoplus_{\alpha}^{p-|p-a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \\
\oplus & \bigoplus_{\beta}^{|p'-b-b'|-1} \left\{ \bigoplus_{\alpha}^{p-|p-a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{|a-a'|-1} 4(\mathcal{R}_{\kappa p, (2 \cdot \kappa') p'}^{\alpha, \beta})_{\mathcal{W}} \right\} \quad (\text{A.11})
\end{aligned}$$

\otimes	$(1,1)_{\mathcal{W}}$	$(2,1)_{\mathcal{W}}$	$(1,2)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{p-1,1})$	$\mathcal{W}(\Delta_{2p-1,1})$	$\mathcal{W}(\Delta_{3p-1,1})$
$(1,1)_{\mathcal{W}}$	$(1,1)_{\mathcal{W}}$	$(2,1)_{\mathcal{W}}$	$(1,2)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{p-1,1})$	$\mathcal{W}(\Delta_{2p-1,1})$	$\mathcal{W}(\Delta_{3p-1,1})$
$(2,1)_{\mathcal{W}}$	$(2,1)_{\mathcal{W}}$	$(1,1)_{\mathcal{W}} \oplus (3,1)_{\mathcal{W}}$	$(2,2)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{p-2,1})$	$\mathcal{W}(\Delta_{2p-2,1})$	$\mathcal{W}(\Delta_{3p-2,1})$
$(1,2)_{\mathcal{W}}$	$(1,2)_{\mathcal{W}}$	$(2,2)_{\mathcal{W}}$	$(1,1)_{\mathcal{W}} \oplus (1,3)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{p-1,2})$	$\mathcal{W}(\Delta_{2p-1,2})$	$\mathcal{W}(\Delta_{3p-1,2})$
$\mathcal{W}(\Delta_{p-1,1})$	$\mathcal{W}(\Delta_{p-1,1})$	$\mathcal{W}(\Delta_{p-2,1})$	$\mathcal{W}(\Delta_{p-1,2})$	$\mathcal{W}(\Delta_{1,1})$	0	0
$\mathcal{W}(\Delta_{2p-1,1})$	$\mathcal{W}(\Delta_{2p-1,1})$	$\mathcal{W}(\Delta_{2p-2,1})$	$\mathcal{W}(\Delta_{2p-1,2})$	0	$(1,1)_{\mathcal{W}}^*$	$\mathcal{W}(\Delta_{3p-1,1})$
$\mathcal{W}(\Delta_{3p-1,1})$	$\mathcal{W}(\Delta_{3p-1,1})$	$\mathcal{W}(\Delta_{3p-2,1})$	$\mathcal{W}(\Delta_{3p-1,2})$	0	$\mathcal{W}(\Delta_{3p-1,1})$	$(1,1)_{\mathcal{W}}^*$

Figure 3: Cayley table of the basic fusion rules for $\text{Conj}[\mathcal{WLM}(p, p')]$ for $p > 2$.

A.2 Some fusion evaluations

The applications of \mathcal{K} , \mathcal{L} , \mathcal{L}^2 , \mathcal{M} and \mathcal{M}^2 appearing in (3.27) read

$$\begin{aligned}
\mathcal{K}[(a, b)_{\mathcal{W}} \otimes (a', b')_{\mathcal{W}}] &= \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} \mathcal{W}(\Delta_{3p-i,j}) \oplus \bigoplus_{\alpha}^{a+a'-p-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (\mathcal{R}_{2p,j}^{\alpha,0})_{\mathcal{W}} \\
&\quad \oplus \bigoplus_{\beta}^{b+b'-p'-1} \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} (\mathcal{R}_{i,2p'}^{0,\beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+a'-p-1} \bigoplus_{\beta}^{b+b'-p'-1} (\mathcal{R}_{2p,p'}^{\alpha,\beta})_{\mathcal{W}} \\
\mathcal{L}[(a, b)_{\mathcal{W}} \otimes (a', b')_{\mathcal{W}}] &= \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} \mathcal{W}(\Delta_{2p-i,j}) \oplus \bigoplus_{\alpha}^{a+a'-p-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (\mathcal{R}_{p,j}^{\alpha,0})_{\mathcal{W}} \\
&\quad \oplus \bigoplus_{\beta}^{b+b'-p'-1} \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} (\mathcal{R}_{i,p'}^{0,\beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+a'-p-1} \bigoplus_{\beta}^{b+b'-p'-1} (\mathcal{R}_{p,p'}^{\alpha,\beta})_{\mathcal{W}} \\
\mathcal{L}^2[(a, b)_{\mathcal{W}} \otimes (a', b')_{\mathcal{W}}] &= \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (i, j)_{\mathcal{W}}^* \oplus \bigoplus_{\alpha}^{a+a'-p-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} (\mathcal{R}_{p,j}^{\alpha,0})_{\mathcal{W}} \\
&\quad \oplus \bigoplus_{\beta}^{b+b'-p'-1} \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} (\mathcal{R}_{i,p'}^{0,\beta})_{\mathcal{W}} \oplus \bigoplus_{\alpha}^{a+a'-p-1} \bigoplus_{\beta}^{b+b'-p'-1} (\mathcal{R}_{p,p'}^{\alpha,\beta})_{\mathcal{W}} \\
\mathcal{M}[(a, b)_{\mathcal{W}} \otimes (a', b')_{\mathcal{W}}] &= \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} \mathcal{W}(\Delta_{p-i,j}) \\
\mathcal{M}^2[(a, b)_{\mathcal{W}} \otimes (a', b')_{\mathcal{W}}] &= \bigoplus_{i=|a-a'|+1, \text{ by } 2}^{p-|p-a-a'|-1} \bigoplus_{j=|b-b'|+1, \text{ by } 2}^{p'-|p'-b-b'|-1} \mathcal{W}(\Delta_{i,j}) \tag{A.12}
\end{aligned}$$

A.3 Basic fusion rules

For $p > 2$, the basic fusion rules of $\text{Conj}[\mathcal{WLM}(p, p')]$ (3.33) are summarized in Figure 3, where $(3, 1)_{\mathcal{W}} = \mathcal{W}(\Delta_{3,1}) \in (\mathcal{J}_{3,p'}^{\text{out}})_{\mathcal{W}}$ for $p = 3$. The basic fusion rules for $p = 2$ are given in Figure 4, where $(2, 1)_{\mathcal{W}} = \mathcal{W}(\Delta_{2,1}) \in (\mathcal{J}_{2,p'}^{\text{out}})_{\mathcal{W}}$ and, for $p' = 3$, $(1, 3)_{\mathcal{W}} = \mathcal{W}(\Delta_{1,3}) \in (\mathcal{J}_{2,3}^{\text{out}})_{\mathcal{W}}$.

\otimes	$(1, 1)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{2,1})$	$(1, 2)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{1,1})$	$\mathcal{W}(\Delta_{3,1})$	$\mathcal{W}(\Delta_{5,1})$
$(1, 1)_{\mathcal{W}}$	$(1, 1)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{2,1})$	$(1, 2)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{1,1})$	$\mathcal{W}(\Delta_{3,1})$	$\mathcal{W}(\Delta_{5,1})$
$\mathcal{W}(\Delta_{2,1})$	$\mathcal{W}(\Delta_{2,1})$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$\mathcal{W}(\Delta_{2,2})$	0	$\mathcal{W}(\Delta_{2,1})$	$\mathcal{W}(\Delta_{4,1})$
$(1, 2)_{\mathcal{W}}$	$(1, 2)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{2,2})$	$(1, 1)_{\mathcal{W}} \oplus (1, 3)_{\mathcal{W}}$	$\mathcal{W}(\Delta_{1,2})$	$\mathcal{W}(\Delta_{3,2})$	$\mathcal{W}(\Delta_{5,2})$
$\mathcal{W}(\Delta_{1,1})$	$\mathcal{W}(\Delta_{1,1})$	0	$\mathcal{W}(\Delta_{1,2})$	$\mathcal{W}(\Delta_{1,1})$	0	0
$\mathcal{W}(\Delta_{3,1})$	$\mathcal{W}(\Delta_{3,1})$	$\mathcal{W}(\Delta_{2,1})$	$\mathcal{W}(\Delta_{3,2})$	0	$(1, 1)_{\mathcal{W}}^*$	$\mathcal{W}(\Delta_{5,1})$
$\mathcal{W}(\Delta_{5,1})$	$\mathcal{W}(\Delta_{5,1})$	$\mathcal{W}(\Delta_{4,1})$	$\mathcal{W}(\Delta_{5,2})$	0	$\mathcal{W}(\Delta_{5,1})$	$(1, 1)_{\mathcal{W}}^*$

Figure 4: Cayley table of the basic fusion rules for $\text{Conj}[\mathcal{WLM}(2, p')]$.

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